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On the first hitting time density for a reducible diffusion process

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In this paper, we study the classical problem of the first hitting time density to a moving boundary for a diffusion process, which satisfies the Cherkasov condition, and hence, can be reduced to a standard Wiener process. We give two complementary (forward and backward) formulations of this problem and provide semi-analytical solutions for both. By using the method of heat potentials, we show how to reduce these problems to linear Volterra integral equations of the second kind. For small values of t, we solve these equations analytically by using Abel equation approximation; for larger t we solve them numerically. We illustrate our method with representative examples, including Ornstein–Uhlenbeck processes with both constant and time-dependent coefficients. We provide a comparison with other known methods for finding the hitting density of interest, and argue that our method has considerable advantages and provides additional valuable insights. We also show applications of the problem and our method in various areas of financial mathematics.

Keywords: First hitting time density; Cherkasov condition; Method of heat potentials; Volterra integral equation; Abel integral equation; Ornstein–Uhlenbeck process; Pairs trading

1. Introduction

Computation of the first hitting time density for a diffusion process is a long-standing problem, which is still a subject of active research. Particular examples are the hitting problem for a Wiener process to a curvilinear boundary and the hitting problem for an Ornstein–Uhlenbeck process with a straight boundary. An abstract approach applicable to this problem has been found by Fortet (1943); Fortet's equation can be viewed as a variant of the Einstein–Smoluchowski equation (Einstein 1905, Von Smoluchowski 1906). A general overview can be found in Breiman (1967), Horowitz (1985), and Borodin and Salminen (2012). More recently, Di Nardo *et al.* (2001) developed numerical approaches for first passage problems for a general Gauss–Markov process. The authors reduced the problem to an integral equation of Fortet type and solved it numerically. In parallel, Daniels (1996), Novikov *et al.* (1999), and Pötzelberger and Wang (2001) derived a formula for a piece-wise linear barrier and used it to develop a numerical method for general curved boundaries by virtue of their piece-wise linear approximation.

For certain choices of diffusion processes and the corresponding barriers, analytical solutions have been found. For instance, for a standard Wiener process, it is easy to derive a closed form formula for a linear boundary. A square-root boundary is much harder to handle, but Shepp (1967) and Novikov (1971) managed to describe the corresponding density and moments expansion in terms of special functions (parabolic cylinder functions and Hermite polynomials). Attempts to find an analytical result for an Ornstein–Uhlenbeck (OU) process have been made since 1998 when Leblanc and Scaillet (1998) first proposed a closed-form formula, which, as it tuned out, contained an error. Two years later, Leblanc *et al.* (2000) published a correction on the paper; unfortunately, the correction itself was erroneous as well. The authors thought that the density in question can be expressed as the Laplace transform of a functional of a three-dimensional Bessel bridge, but Göing-Jaeschke and Yor (2003) noticed that they had incorrectly used a spatial homogeneity property for such a bridge. It is worth noting that the same conclusion can be obtained much more straightforwardly by calculating the CDF and showing that it asymptotically exceeds unity for some values of parameters.

Given the time-independence of the hitting density problem for an OU process with *flat* boundary, it is natural to attack

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this problem by using the Laplace transform, see, e.g. Ricciardi and Sato (1988), Linetsky (2004), and Alili *et al.* (2005). In particular, Alili *et al.* (2005) gave several complementary representations: a series of parabolic cylinder functions and its derivatives, an indefinite integral representation via special functions, and a Bessel bridge representation. The first approach is based on inverting the Laplace transform, which is computed analytically and results in a series representation involving parabolic cylinder functions and their derivatives. The second approach is based on the cosine transform and its inverse and produces an indefinite integral involving some special functions and is computed using the trapezoidal rule. The third approach gives a representation of the density via an expectation of a function of the three-dimensional Bessel bridge, which can be computed using the Monte Carlo method. Linetsky (2004) gave an analytical representation via the relevant Sturm–Liouville eigenfunction expansion. The coefficients were found as a solution of a nonlinear equation, which involved nonlinear special (Hermite) functions. The main disadvantage of this approach is that it requires one to solve a separate nonlinear equation to find each new coefficient.

A recent paper by Jiang *et al.* (2019) studied the joint distribution and the multivariate survival functions for the maxima of the OU process as well as for general Markov processes with time-dependent barriers. The authors adopted the expansion results described above; for the case of time-dependent coefficients and barrier, they approximated these by piece-wise constant functions and developed a numerical method by solving the problem on each interval.

An important breakthrough was achieved by Martin *et al.* (2015), who solved a nonlinear Fokker–Planck equation with a steady-state solution by representing it as an infinite product rather than as usual an infinite sum. The PDE, which corresponds to the transition probability of the Ornstein–Uhlenbeck process, belongs to the class of equations described in Martin *et al.* (2015). In principle, the results of that paper can be modified to compute the first time hitting density. An advantage of this method is that it allows quantifying the errors; thereby controlling the number of terms required to reach a given precision. More recently, Martin *et al.* (2019) developed a short- and long-time asymptotic expansion for the OU and other mean-reverting processes based on similar principles.

All the methods described above require substantial numerical computations, and for some of them, the convergence rate is unknown. Moreover, most of them are difficult to implement. For instance, Cheridito and Xu (2015) preferred the Crank–Nicolson method to other known analytical methods, because it is easier to implement.

In this paper, which is an extension and improvement of Lipton and Kaushansky (2018), we develop a fast and easily implementable semi-analytical method to compute the first time hitting density of a general diffusion process of the Cherkasov type. We give particular examples for the cases of a Wiener process and time-dependent Ornstein–Uhlenbeck processes.

The main idea is as follows: after an appropriate change of variables, the corresponding PDE becomes a heat equation with a moving boundary, which can be solved by using the method of heat potentials (Lipton 2001, Section 12.2.3, pp. 462–467). As usual, the method of heat potentials leads to a Volterra equation of the second kind, which can be solved numerically, see Lipton *et al.* (2018) and Lipton and Kaushansky (2018). As a result, we obtain recursive formulas to compute the corresponding hitting density. It is also worth pointing out that for the case of an Ornstein–Uhlenbeck process, our method is able to deal with time-dependent coefficients, while the methods which are based on the Laplace transform are only able to deal with the case of constant coefficients.

Below we discuss several important financial applications of our method, including barrier options pricing, pairs trading, and credit risk modeling. The latter results generalize the work of Hyer *et al.* (1999), Hull and White (2001), and Avellaneda and Zhu (2001) who analyzed a structural default model and developed numerical methods to find the default boundary knowing the corresponding default probabilities.

The rest of the paper is organized as follows: in Section 2 we formulate the problem and eliminate parameters using a change of variables; in Section 3 we give our main results; in Section 4, we illustrate our results on several important and informative examples: Wiener process, Geometric Brownian motion, Ornstein–Uhlenbeck process; in Section 5 we consider a numerical solution for the corresponding Volterra equation as well as a solution as an approximation by an Abel equation; in Section 6 we give numerical illustrations and compare the methods; in Section 7, we show important applications of our method in financial mathematics; in Section 8 we conclude.

2. Problem formulation and initial transformations

2.1. *First hitting time density for a Wiener process*

Consider the distribution of the hitting time of a Wiener process with $W_0 = z$ of a given time-dependent barrier $b(t)$:

$$s = \inf\{t : W_t = b(t)\}, \quad z \neq b(0).$$

Forward problem. Using Feynman–Kac formula, the transition probability density satisfies

$$\begin{aligned} p_t(t,x;z) &= \tfrac{1}{2} p_{xx}(t,x;z), \\ p(0,x;z) &= \delta(x-z), \\ p(t,b(t);z) &= 0. \end{aligned}$$

This distribution is given by

$$g(t,z) = \frac{\varpi}{2} p_x(t,b(t);z), \tag{1}$$

where

$$\varpi = \operatorname{sgn}(z - b(0)).$$

Backward problem. Alternatively, we can solve the corresponding backward problem for the cumulative hitting probability $G(t,T,z)$. We can express it as

$$G(t,T,z)=\mathbb{E}[\mathbb{1}_{\{\exists s\in[t,T]:W_s=b(s)\}}\mid W_t=z]. \tag{2}$$

Then, using Feynman–Kac formula, we can write corresponding PDE

$$\begin{aligned} &G_t\left(t,T,z\right)+\tfrac{1}{2}G_{zz}\left(t,T,z\right)=0,\\ &G\left(T,T,z\right)=0,\\ &G\left(t,T,b(t)\right)=1. \end{aligned} \tag{3}$$

Thus, the first hitting density is

$$g\left(T,z\right)=G_T\left(0,T,z\right).$$

2.2. *First hitting time density of a Cherkasov process*

Now consider a general diffusion satisfying the so-called Cherkasov's condition which guarantees that it can be transformed into the standard Wiener process.

$$dX_t=\delta\left(t,X_t\right)\mathrm{d}t+\sigma\left(t,X_t\right)\mathrm{d}W_t,\quad X_0=z, \tag{4}$$

which has been studied in Cherkasov (1957), Ricciardi (1976), and Bluman (1980). The applications of Cherkasov conditions in financial mathematics were studied in Lipton (2001) in Section 4.2 and Lipton (2018) in Chapter 9.

We wish to calculate the distribution of the hitting time of this process of a given time-dependent barrier $b(t)$:

$$s=\inf\left\{t:X_t=b(t)\right\},\quad z\neq b(0).$$

Introduce

$$\begin{aligned} \beta\left(t,x\right)&=\sigma\left(t,x\right)\int^x\frac{1}{\sigma\left(t,y\right)}\mathrm{d}y,\\ \gamma\left(t,x\right)&=2\delta\left(t,x\right)-\sigma\left(t,x\right)\sigma_x\left(t,x\right)\\ &\quad-2\sigma\left(t,x\right)\int^x\frac{\sigma_t\left(t,y\right)}{\sigma^2\left(t,y\right)}\mathrm{d}y, \end{aligned}$$

where the lower limit of the integral is chosen as convenient. Let

$$P\left(t,x\right)=\begin{vmatrix}\beta\left(t,x\right) & \gamma\left(t,x\right)\\ \beta_x\left(t,x\right) & \gamma_x\left(t,x\right)\end{vmatrix},$$

$$Q\left(t,x\right)=\begin{vmatrix}\sigma\left(t,x\right) & \gamma\left(t,x\right)\\ \sigma_x\left(t,x\right) & \gamma_x\left(t,x\right)\end{vmatrix},$$

$$R\left(t,x\right)=\begin{vmatrix}\sigma\left(t,x\right) & \beta\left(t,x\right) & \gamma\left(t,x\right)\\ \sigma_x\left(t,x\right) & \beta_x\left(t,x\right) & \gamma_x\left(t,x\right)\\ \sigma_{xx}\left(t,x\right) & \beta_{xx}\left(t,x\right) & \gamma_{xx}\left(t,x\right)\end{vmatrix}.$$

Assume that so-called Cherkasov condition is satisfied, so that

$$R\left(t,x\right)\equiv 0.$$

Then we can transform x into the standard Wiener process via the following mapping

$$\tilde{t}=\tilde{t}(t,x)=\int_0^t\Phi^2(u,x)du, \tag{5}$$

$$\tilde{x}=\tilde{x}(t,x)=\Phi(t,x)\frac{\beta\left(t,x\right)}{\sigma\left(t,x\right)}+\frac{1}{2}\int_0^t\Phi(u,x)\frac{P\left(u,x\right)}{\sigma\left(u,x\right)}\mathrm{d}u, \tag{6}$$

where

$$\Phi\left(t,x\right)=\exp\left[-\frac{1}{2}\int_0^t\frac{Q\left(u,x\right)}{\sigma\left(u,x\right)}\mathrm{d}u\right]. \tag{7}$$

In particular,

$$\tilde{z}=\frac{\beta\left(0,z\right)}{\sigma\left(0,z\right)}. \tag{8}$$

The corresponding transition probability density transforms

$$p(t,x;z)=\left|\frac{\partial\tilde{x}(t,x)}{\partial x}\right|\tilde{p}(\tilde{t},\tilde{x};\tilde{z}).$$

Moreover, the boundary transforms to

$$b(t)\to\tilde{b}(\tilde{t}),$$

where

$$\tilde{b}(\tilde{t})=\Phi(t,b(t))\frac{\beta\left(t,b(t)\right)}{\sigma\left(t,b(t)\right)}+\frac{1}{2}\int^t\Phi(u,b(t))\frac{P\left(u,b(t)\right)}{\sigma\left(u,b(t)\right)}\mathrm{d}u. \tag{9}$$

Whenever possible, below we omit tilde for brevity.

3. Main results

In this section, we consider four main theorems, which solve forward and backward problems for a Wiener and a general Cherkasov processes.

3.1. *Forward problem for the Wiener process*

THEOREM 1 *The hitting density* (1) *can be written as*

$$\begin{aligned} g\left(t,z\right)=&\ \frac{1}{2}\int_0^t\frac{\left(1-\frac{\Psi\left(t,t'\right)^2}{\left(t-t'\right)}\right)\Xi\left(t,t'\right)\nu\left(t'\right)-\nu\left(t\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t'\\ &-\frac{1}{\sqrt{2\pi t}}\nu\left(t\right)\\ &+\varpi\left(-b'\left(t\right)\nu\left(t\right)+\frac{\left(z-b\left(t\right)\right)\exp\left(-\frac{\left(z-b(t)\right)^2}{2t}\right)}{2\sqrt{2\pi t^3}}\right). \end{aligned} \tag{10}$$

where

$$\Psi\left(t,t'\right)=b\left(t\right)-b\left(t'\right),\quad t\geq t',$$

$$\Xi\left(t,t'\right)=\begin{cases}\exp\left(-\frac{\Psi\left(t,t'\right)^2}{2\left(t-t'\right)}\right), & t>t'\\ 1, & t=t'\end{cases}.$$

and $\nu(t)$ *is the solution of Volterra integral equation of the second kind*

$$\nu\left(t\right)+\varpi\int_0^t\frac{\Psi\left(t,t'\right)\Xi\left(t,t'\right)\nu\left(t'\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t'=-\frac{\exp\left(-\frac{\left(z-b(t)\right)^2}{2t}\right)}{\sqrt{2\pi t}}. \tag{11}$$

3.2. *Backward problem for Wiener process*

THEOREM 2 *Choose some positive $T > 0$. The CDF* (2) *can be written as*

$$G(0,T,z) = \varpi \int_0^T \frac{\left(z - b\left(T - t'\right)\right)\exp\left(-\frac{(z-b(T-t'))^2}{2(T-t')}\right)\nu\left(t',T\right)}{\sqrt{2\pi\,(T-t')^3}} \mathrm{d}t', \quad (12)$$

where

$$\nu(t,T) + \varpi \int_0^t \times \frac{\Psi\left(T-t, T-t'\right)\Xi(T-t,T-t')\nu\left(t',T\right)}{\sqrt{2\pi\,(t-t')^3}} \mathrm{d}t' = 1. \quad (13)$$

REMARK 3 Theorem 1 is easier to use because it allows one to calculate $g(t)$ in one go, while Theorem 2 requires solving a different equation for every T, except of the trivial case $b(t) \equiv b$. However, Theorem 2 is useful if a particular point in time is of interest, say $T = \infty$.

3.3. *Forward problem for a general Cherkasov process*

In this section, we formulate the result for a general process (4) using the transformations from Section 2.2 and the results of Section 3.1.

THEOREM 4 *The hitting density of a general Cherkasov process can be written as*

$$g(t,z) = \frac{\mathrm{d}\tilde{t}(t,b(t))}{\mathrm{d}t} \tilde{g}(\tilde{t},\tilde{z}), \quad (14)$$

where $\tilde{g}(\tilde{t},\tilde{z})$ is the hitting density of a Wiener process, $\tilde{t}(t,x)$ is defined in (5), *$\tilde{b}(\tilde{t})$ is defined in* (9), *$\tilde{z}(z)$ is defined in* (8), *and*

$$\frac{\mathrm{d}\tilde{t}(t,b(t))}{\mathrm{d}t} = \Phi^2(t,b(t)) + 2b'(t)\int_0^t \Phi(s,b(t))\Phi_x(s,b(t))\,\mathrm{d}s.$$

3.4. *Backward problem for a general Cherkasov process*

As in previous section, we extend the result from a Wiener process to a general Cherkasov process.

THEOREM 5 *Choose some positive $T > 0$. The CDF of the hitting time for* (4) *can be written as*

$$G(T,z) = \tilde{G}(\tilde{T},\tilde{z}), \quad (15)$$

where $\tilde{G}(\tilde{T},\tilde{z})$ is the hitting density of the transformed Wiener process.

3.5. *Proof of Theorem* 1

Consider $\varphi > 0$. To find the hitting density, we use the method of heat potentials (see Tikhonov and Samarskii 1963, pp. 530–535; Lipton 2001, Section 12.2.3, pp. 462–467 for details). As usual, we write

$$p(t,x) = \tilde{H}(t,x,0,z) + q(t,x),$$

where $\tilde{H}(t,x,0,z)$ is the standard heat kernel, while $q(t,x)$ solves the IBVP of the form

$$\begin{aligned} q_t(t,x) &= \tfrac{1}{2}q_{xx}(t,x), \\ q(0,x) &= 0, \\ q(t,b(t)) &= -\tilde{H}(t,b(t),0,z). \end{aligned}$$

Accordingly,

$$q(t,x) = \int_0^t \frac{\left(x - b\left(t'\right)\right)\exp\left(-\frac{(x-b(t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\,(t-t')^3}} \nu\left(t'\right)\mathrm{d}t',$$

where ν is the solution of the Volterra equation of the second kind,

$$\nu(t) + \int_0^t \frac{\Psi(t,t')\Xi\left(t,t'\right)\nu\left(t'\right)}{\sqrt{2\pi\,(t-t')^3}} dt' + \frac{\exp\left(\left(-\frac{(b(t)-z)^2}{2t}\right)\right)}{\sqrt{2\pi t}} = 0.$$

Assuming that $\nu(t)$ is found, we can proceed as follows:

$$p(t,x) = \frac{\exp\left(-(x-z)^2/(2t)\right)}{\sqrt{2\pi t}} + q(t,x), \quad (16)$$

Then, $g(t)$ can be computed using (1) by differentiation of (16) and taking its limit at $b(t)$. We do the necessary computations in Appendix, and arrive at the final formula (10).

3.6. *Proof of Theorem* 2

Consider $\varphi > 0$. Fix $T > 0$. Consider the change of variables $\tau = T - t$ with $\hat{G}(\tau,T,z) = G(T-t,T,z)$ in (3). Then,

$$\begin{aligned} \hat{G}_\tau(\tau,T,z) &= \tfrac{1}{2}\hat{G}_{zz}(\tau,T,z), \\ \hat{G}(0,T,z) &= 0, \\ \hat{G}(\tau,T,b(T-\tau)) &= 1. \end{aligned} \quad (17)$$

We got that $\hat{G}(t,T,z)$ satisfies the heat equation with moving boundary and zero initial condition. As a result, as before, we are able to apply the method of heat potentials. We get

$$\hat{G}(\tau,T,z) = \int_0^\tau \frac{\left(z - b\left(T-t'\right)\right)\exp\left(-\frac{(z-b(T-t'))^2}{2(T-t')}\right)\nu\left(t',T\right)}{\sqrt{2\pi\,(T-t')^3}} \mathrm{d}t',$$

where

$$\nu(t,T)+\int_0^t \frac{\left(\Psi\left(T-t,T-t'\right)\right)\Xi(T-t,T-t')\nu\left(t',T\right)}{\sqrt{2\pi\left(t-t'\right)^3}}dt' = 1.$$

As $G(0,T,z)=\hat{G}(T,T,z)$, we immediately get

$$G(0,T,z) = \int_0^{\mathrm{T}} \frac{\left(z-b\left(T-t'\right)\right)\exp\left(-\frac{\left(z-b\left(T-t'\right)\right)^2}{2(T-t')}\right)\nu\left(t',T\right)}{\sqrt{2\pi\left(T-t'\right)^3}}dt'.$$

4. Representative examples

4.1. Linear boundary

This is a classical example, and considered, for example, in Lipton (2001), among many others. Here, we provide an alternative derivation using the theory we developed before.

When the boundary is linear,

$$b(t)=\mu_0+\mu_1 t, \quad z>\mu_0,$$

the pair $(\nu(t),g(t))$ can be calculated explicitly by skillfully using a combination of the direct and inverse Laplace transforms. In other cases, it has to be calculated numerically.

Since

$$\Psi\left(t,t'\right)=\mu_1\left(t-t'\right), \quad \Xi\left(t,t'\right)=\exp\left(-\frac{\mu_1^2\left(t-t'\right)}{2}\right),$$

we have

$$\nu(t)+\mu_1\int_0^t \frac{\exp\left(-\frac{\mu_1^2(t-t')}{2}\right)\nu\left(t'\right)}{\sqrt{2\pi\left(t-t'\right)}}dt' = -\frac{\exp\left(-\frac{(z-\mu_0-\mu_1 t)^2}{2t}\right)}{\sqrt{2\pi t}}, \tag{18}$$

$$g(t) = -\left(\mu_1+\frac{1}{\sqrt{2\pi t}}\right)\nu(t) + \frac{1}{2}\int_0^t \frac{\left(1-\mu_1^2\left(t-t'\right)\right)\exp\left(-\frac{\mu_1^2(t-t')}{2}\right)\nu\left(t'\right)-\nu(t)}{\sqrt{2\pi\left(t-t'\right)^3}}dt' + \frac{(z-\mu_0-\mu_1 t)\exp\left(-\frac{(z-\mu_0-\mu_1 t)^2}{2t}\right)}{2\sqrt{2\pi t^3}}. \tag{19}$$

By rearranging terms and using integration by parts, we can rewrite (19) as follows

$$g(t)=-g_1(t)+g_2(t), \tag{20}$$

$$g_1(t)=\int_0^t \frac{\exp\left(-\frac{\mu_1^2(t-t')}{2}\right)\dot{\nu}\left(t'\right)}{\sqrt{2\pi\left(t-t'\right)}}dt, \tag{21}$$

$$g_2(t)=\frac{(z-\mu_0+\mu_1 t)\exp\left(-\frac{(z-\mu_0-\mu_1 t)^2}{2t}\right)}{2\sqrt{2\pi t^3}}. \tag{22}$$

The Laplace transform of (18), (21) yields

$$\left(1+\frac{\mu_1}{\sqrt{2s+\mu_1^2}}\right)\hat{\nu}(s) = -\frac{\exp\left(-(z-\mu_0)\left(\sqrt{2s+\mu_1^2}-\mu_1\right)\right)}{\sqrt{2s+\mu_1^2}},$$

$$\hat{g}_1(s)=\frac{s\hat{\nu}(s)}{\sqrt{2s+\mu_1^2}}.$$

Accordingly, by taking the inverse Laplace transform,

$$\nu(t) = -\frac{\exp\left(-\frac{(z-\mu_0-\mu_1 t)^2}{2t}\right)}{\sqrt{2\pi t}} + \mu_1\exp\left(2(z-\mu_0)\mu_1\right)N\left(-\frac{z-\mu_0}{\sqrt{t}}-\mu_1\sqrt{t}\right), \tag{23}$$

and

$$g_1(t) = -\frac{(z-\mu_0)\exp\left(-\frac{(z-\mu_0-\mu_1 t)^2}{2t}\right)}{2\sqrt{2\pi t^3}} + \frac{\mu_1\exp\left(-\frac{(z-\mu_0-\mu_1 t)^2}{2t}\right)}{2\sqrt{2\pi t}}.$$

Finally, we arrive at

$$g(t)=\frac{(z-\mu_0)\exp\left(-\frac{(z-\mu_0-\mu_1 t)^2}{2t}\right)}{\sqrt{2\pi t^3}}, \tag{24}$$

as expected. When $\mu_1=0$, (24) trivially follows from the method of images.

4.2. Time-dependent geometric Brownian motion

Consider a time-dependent geometric Brownian motion. Then

$$\delta(t,x)=\delta(t)x, \quad \sigma(t,x)=\sigma(t)x,$$

$$\beta(t,x)=x\ln x,$$

$$\gamma(t,x)=\left(2\delta(t)-\sigma^2(t)\right)x-2\frac{\sigma_t(t)}{\sigma(t)}x\ln x,$$

$$P(t,x)=\begin{vmatrix} x\ln x & \left(2\delta(t)-\sigma^2(t)\right)x-2\frac{\sigma_t(t)}{\sigma(t)}x\ln x \\ \ln x+1 & \left(2\delta(t)-\sigma^2(t)\right)-2\frac{\sigma_t(t)}{\sigma(t)}(\ln x+1) \end{vmatrix}$$

$$= -\left(2\delta(t) - \sigma^2(t)\right)x,$$

$$Q(t,x) = \begin{vmatrix} \sigma(t)x & \left(2\delta(t) - \sigma^2(t)\right)x - 2\frac{\sigma_t(t)}{\sigma(t)}x\ln x \\ \sigma(t) & \left(2\delta(t) - \sigma^2(t)\right) - 2\frac{\sigma_t(t)}{\sigma(t)}(\ln x + 1) \end{vmatrix}$$

$$= -2\sigma_t(t)x,$$

$$R(t,x) = \begin{vmatrix} \sigma(t)x & x\ln x & \left(2\delta(t) - \sigma^2(t)\right)x - 2\frac{\sigma_t(t)}{\sigma(t)}x\ln x \\ \sigma(t) & \ln x + 1 & \left(2\delta(t) - \sigma^2(t)\right) - 2\frac{\sigma_t(t)}{\sigma(t)}(\ln x + 1) \\ 0 & \frac{1}{x} & -2\frac{\sigma_t(t)}{\sigma(t)x} \end{vmatrix}$$

$$\equiv 0.$$

Accordingly,

$$\tilde{t} = \int_0^t \exp\left[2\int_0^u \frac{\sigma_t(s)}{\sigma(s)}\mathrm{d}s\right]\mathrm{d}u$$

$$= \int_0^t \exp\left[2\ln\sigma(u)\right]\mathrm{d}u = \int_0^t \sigma^2(u)\,\mathrm{d}u,$$

$$\tilde{x} = \ln(x) - \int_0^t \left(\delta(u) - \frac{1}{2}\sigma^2(u)\right)\mathrm{d}u,$$

$$\tilde{b}(t) = \ln b(t) - \tilde{\delta}(\tilde{t}) - \frac{\tilde{t}}{2},$$

$$\tilde{z} = \ln z,$$

where

$$\tilde{\delta}(\tilde{t}) = \int_0^t \delta(u)\,\mathrm{d}u.$$

Naturally, this is in agreement with Girsanov's theorem.

Thus, using Theorem 4, we can express the solution:

$$g(t,z) = \sigma^2(t)\left[\frac{1}{2}\int_0^{\tilde{t}(t)} \frac{\left(1 - \frac{\tilde{\Psi}\left(\tilde{t}(t),\tilde{t}'\right)^2}{\left(\tilde{t}(t)-\tilde{t}'\right)}\right)\tilde{\Xi}\left(\tilde{t}(t),\tilde{t}'\right)}{\sqrt{2\pi\left(\tilde{t}(t)-\tilde{t}'\right)^3}}\frac{}{}\tilde{\nu}\left(\tilde{t}'\right) - \tilde{\nu}\left(\tilde{t}(t)\right)\,\mathrm{d}\tilde{t}' - \frac{1}{\sqrt{2\pi\tilde{t}(t)}}\tilde{\nu}\left(\tilde{t}(t)\right) + \varpi\left(-\tilde{b}'\left(\tilde{t}(t)\right)\tilde{\nu}\left(\tilde{t}(t)\right) + \frac{\left(\ln z - \tilde{b}\left(\tilde{t}(t)\right)\right)\exp\left(-\frac{\left(\ln z - \tilde{b}\left(\tilde{t}(t)\right)\right)^2}{2t}\right)}{2\sqrt{2\pi\tilde{t}(t)^3}}\right)\right], \tag{25}$$

where

$$\tilde{b}'(\tilde{t}(t)) = \frac{1}{\sigma^2(t)}\left[\frac{b'(t)}{b(t)} - \delta(t)\right] - \frac{1}{2}.$$

4.3. *Time-dependent OU process*

Consider a time-dependent Ornstein–Uhlenbeck process. Then

$$\delta(t,x) = \lambda(t)(\theta(t) - x),$$

$$\sigma(t,x) = \sigma(t),$$

$$\beta(t,x) = x,$$

$$\gamma(t,x) = 2\lambda(t)(\theta(t) - x) - 2\frac{\sigma_t(t)}{\sigma(t)}x,$$

$$P(t,x) = \begin{vmatrix} x & 2\lambda(t)(\theta(t) - x) - 2\frac{\sigma_t(t)}{\sigma(t)}x \\ 1 & -2\lambda(t) - 2\frac{\sigma_t(t)}{\sigma(t)} \end{vmatrix}$$

$$= -2\lambda(t)\theta(t),$$

$$Q(t,x) = \begin{vmatrix} \sigma(t) & 2\lambda(t)(\theta(t) - x) - 2\frac{\sigma_t(t)}{\sigma(t)}x \\ 0 & -2\lambda(t) - 2\frac{\sigma_t(t)}{\sigma(t)} \end{vmatrix}$$

$$= -2\lambda(t)\sigma(t) - 2\sigma_t(t),$$

$$R(t,x) = \begin{vmatrix} \sigma(t) & x & 2\lambda(t)(\theta(t) - x) - 2\frac{\sigma_t(t)}{\sigma(t)}x \\ 0 & 1 & -2\lambda(t) - 2\frac{\sigma_t(t)}{\sigma(t)} \\ 0 & 0 & 0 \end{vmatrix}$$

$$\equiv 0.$$

Accordingly,

$$\tilde{t} = \int_0^t e^{2\Lambda(u)}\sigma^2(u)\,\mathrm{d}u,$$

$$\tilde{x} = e^{\Lambda(t)}x - \int_0^t e^{\Lambda(u)}\lambda(u)\theta(u)\,\mathrm{d}u,$$

$$\tilde{b}(\tilde{t}) = e^{\Lambda(t)}b(t) - \int_0^t e^{\Lambda(u)}\lambda(u)\theta(u)\,\mathrm{d}u,$$

$$\tilde{z} = z,$$

where

$$\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u.$$

Then, using Theorem 4, we have

$$g(t,z) = e^{2\Lambda(t)}\sigma^2(t)\left[\frac{1}{2}\int_0^{\tilde{t}(t)} \frac{\left(1-\frac{\tilde{\Psi}(\tilde{t}(t),\tilde{t}')^2}{(\tilde{t}(t)-\tilde{t}')}\right)\tilde{\Xi}\left(\tilde{t}(t),\tilde{t}'\right)}{\sqrt{2\pi\left(\tilde{t}(t)-\tilde{t}'\right)^3}} \tilde{\nu}\left(\tilde{t}'\right)-\tilde{\nu}\left(\tilde{t}(t)\right) \mathrm{d}\tilde{t}' - \frac{1}{\sqrt{2\pi\tilde{t}(t)}}\tilde{\nu}\left(\tilde{t}(t)\right) + \varpi\left(-\tilde{b}'\left(\tilde{t}(t)\right)\tilde{\nu}\left(\tilde{t}(t)\right) + \frac{\left(\tilde{z}-\tilde{b}\left(\tilde{t}(t)\right)\right)\exp\left(-\frac{\left(\tilde{z}-\tilde{b}(\tilde{t}(t))\right)^2}{2t}\right)}{2\sqrt{2\pi\tilde{t}(t)^3}}\right)\right].$$

4.4. Alternative transformation for the OU process

In this section, we develop an alternative transformation. First, we show how to transform an OU process with constant coefficients into the standard Ornstein–Uhlenbeck process. Second, we demonstrate how to transform any time-dependent process into the standard Wiener process.

We define the standard time-independent OU process

$$\mathrm{d}\bar{x} = -\bar{x}\,\mathrm{d}\bar{t} + \mathrm{d}W_{\bar{t}}.$$

Transformation of an OU process with constant coefficients to the standard OU process. Consider

$$\mathrm{d}X_t = \lambda(\theta - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t.$$

By introducing new variables

$$\bar{t} = \lambda t, \quad \bar{X} = \frac{\sqrt{\lambda}}{\sigma}(X-\theta), \quad \bar{z} = \frac{\sqrt{\lambda}}{\sigma}(z-\theta),$$
$$\bar{b}(t) = \frac{\sqrt{\lambda}}{\sigma}(b(t)-\theta),$$

we can rewrite the problem as follows

$$\mathrm{d}\bar{X}_{\bar{t}} = -\bar{X}_{\bar{t}}\,\mathrm{d}\bar{t} + \mathrm{d}W_{\bar{t}},$$
$$\bar{X}_0 = \bar{z},$$
$$\bar{s} = \inf\left\{\bar{t} : \bar{X}_{\bar{t}} = \bar{b}(t)\right\},$$

Below bars are omitted for brevity whenever possible.

Transformation of a time-dependent OU process to the standard OU process. Now consider the time-dependent process,

$$\mathrm{d}X_t = \lambda(t)\left(\theta(t) - X_t\right)\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t,$$
$$X_0 = z.$$

We transform it into the standard process in two steps. First, we introduce

$$\bar{x} = p\left(\bar{t}\right)x + q\left(\bar{t}\right),$$

and notice that

$$\mathrm{d}\bar{x} = \left(p\lambda\theta + q' + \frac{\left(p\lambda - p'\right)q}{p} - \frac{\left(p\lambda - p'\right)}{p}y\right)\mathrm{d}t + p\sigma\,\mathrm{d}W_t.$$

We impose the following constraints:

$$\lambda\theta p + q' + \frac{\left(\lambda p - p'\right)q}{p} = 0, \quad q(0) = q_0,$$
$$\frac{\left(\lambda p - p'\right)}{\sigma^2 p^3} = 1, \quad p(0) = p_0,$$

or, equivalently,

$$q' = -\sigma^2 p^2 q - \lambda\theta p, \quad q(0) = q_0,$$
$$p' = \lambda p - \sigma^2 p^3, \quad p(0) = p_0. \tag{26}$$

Provided that system (26) is solved, we can introduce

$$\bar{t} = \int_0^t \sigma^2 p^2\,\mathrm{d}t,$$

and notice that $\bar{x}(\bar{t})$ is governed by the standard OU process.

In general, one can solve the system (26) in two steps. First, since the first equation is a linear inhomogeneous equation for q, it can be trivially solved provided that p is known. Second, the second equation is a Bernoulli equation for p, which can be solved via the standard substitution

$$r = p^{-2}, \quad p = r^{-1/2}.$$

The corresponding equation for r is a linear inhomogeneous equation, which can be trivially solved. Indeed,

$$-\frac{r'}{2r^{3/2}} = \frac{\lambda}{r^{1/2}} - \frac{\sigma^2}{r^{3/2}},$$

or,equivalently,

$$r' = -2\lambda r + 2\sigma^2.$$

Thus, r, p, q can all be found by quadratures. Finally, $\bar{t}$ can be found by quadrature as well since the corresponding integrand is known

$$\bar{t} = \int_0^t \sigma^2 p^2\,\mathrm{d}t' = \int_0^t \frac{\sigma^2}{r}\mathrm{d}t'.$$

In the following, we omit bar for brevity.

Specifically,

$$\begin{aligned} r(t) &= \mathrm{e}^{-2(\Lambda(t)-M(t))}, \\ p(t) &= p_0\,\mathrm{e}^{\Lambda(t)-M(t)}, \\ q(t) &= \mathrm{e}^{-M(t)}\left(-p_0\int_0^t \lambda(u)\,\theta(u)\,\mathrm{e}^{\Lambda(u)}\mathrm{d}u + q_0\right), \\ \bar{t}(t) &= M(t). \end{aligned} \tag{27}$$

Here

$$\begin{aligned} \Lambda(t) &= \int_0^t \lambda(u)\,\mathrm{d}u, \\ M(t) &= \frac{1}{2}\ln\left(2p_0^2\int_0^t \sigma^2(u)\,\mathrm{e}^{2\Lambda(u)}\mathrm{d}u + 1\right). \end{aligned}$$

It is easy to see that for constant parameters, expressions (27) produce correct results.

Remark 6 It is worth noting that considering $\theta(t)$ as a periodic function, and other parameters as constants, one can reduce an important problem in neuroscience to the standard OU process. It will be reported elsewhere.

Transformation of the standard OU process to a Wiener process (Forward). To calculate the density of the hitting time distribution $g(t,z)$, we need to solve the following forward problem

$$\begin{aligned} p_t(t,x;z) &= p(t,x;z) + xp_x(t,x;z) + \tfrac{1}{2}p_{xx}(t,x;z), \\ p(0,x;z) &= \delta(x-z), \\ p(t,b(t);z) &= 0. \end{aligned} \tag{28}$$

This distribution is given by

$$g(t,z) = \tfrac{1}{2}\varpi p_x(t,b(t);z). \tag{29}$$

For simplicity, consider the case $z > b(0)$.

We wish to transform an initial boundary value problem (IBVP) (28) into the standard IBVP for a heat equation with a moving boundary. To this end, we introduce new independent and dependent variables as follows:

$$q(\tau,\xi) = \mathrm{e}^{-t}p(t,x), \quad \tau = \mathrm{e}^t\sinh(t), \quad \xi = \mathrm{e}^t x.$$

We get

$$\begin{aligned} q_\tau(\tau,\xi) &= \frac{1}{2}q_{\xi\xi}(\tau,\xi), \\ q(0,\xi) &= \delta(\xi - z), \\ q(\tau,\beta(\tau)) &= 0, \\ \beta(\tau) &= \sqrt{2\tau+1}\tilde{b}(t), \end{aligned}$$

where $\tilde{b}(\tau) = b(t(\tau))$.

It is clear that $0 \le \tau < \infty$ and $\mathrm{e}^t = \sqrt{2\tau+1}$.

Then, using Theorem 1, the solution is given by (10) and (11).

Assuming that the boundary is flat,

$$\begin{aligned} \frac{\beta(\tau)-\beta(\tau')}{\tau-\tau'} &= \frac{b\left(\sqrt{2\tau+1}-\sqrt{2\tau'+1}\right)}{\tau-\tau'} \\ &= \frac{2b}{\sqrt{2\tau+1}+\sqrt{2\tau'+1}}, \end{aligned}$$

so that (11) can be written in the form

$$\begin{aligned} &\nu(\tau) + \sqrt{\frac{2}{\pi}}b\int_0^\tau \frac{\exp\left(-b^2\frac{\left(\sqrt{2\tau+1}-\sqrt{2\tau'+1}\right)}{\left(\sqrt{2\tau+1}+\sqrt{2\tau'+1}\right)}\right)\nu(\tau')}{\left(\sqrt{2\tau+1}+\sqrt{2\tau'+1}\right)\sqrt{\tau-\tau'}}\mathrm{d}\tau' \\ &+ \frac{\exp\left(-\frac{\left(\sqrt{2\tau+1}b-z\right)^2}{2\tau}\right)}{\sqrt{2\pi\tau}} = 0. \end{aligned} \tag{30}$$

Hence,

$$\begin{aligned} g(t) = &-\frac{\left(\mathrm{e}^t b - z\right)\exp\left(-\frac{\left(\mathrm{e}^t b-z\right)^2}{\left(\mathrm{e}^{2t}-1\right)} + 2t\right)}{\sqrt{\pi\left(\mathrm{e}^{2t}-1\right)^3}} \\ &- \left(\mathrm{e}^t b + \frac{\mathrm{e}^{2t}}{\sqrt{\pi\left(\mathrm{e}^{2t}-1\right)}}\right)\nu(t) \\ &+ \frac{\mathrm{e}^{2t}}{\sqrt{8\pi}}\int_0^\tau \frac{\left(1-2b^2\frac{\left(\sqrt{2\tau+1}-\sqrt{2\tau'+1}\right)}{\left(\sqrt{2\tau+1}+\sqrt{2\tau'+1}\right)}\right)\exp\left(-b^2\frac{\left(\sqrt{2\tau+1}-\sqrt{2\tau'+1}\right)}{\left(\sqrt{2\tau+1}+\sqrt{2\tau'+1}\right)}\right)\nu(\tau') - \nu(\tau)}{\sqrt{(\tau-\tau')^3}}\mathrm{d}\tau'. \end{aligned} \tag{31}$$

We can rewrite (30) in an alternative way. Let $\theta = \sqrt{2\tau+1} - 1$, $\theta' = \sqrt{2\tau'+1} - 1$, $0 \le \theta' \le \theta < \infty$. Then

$$\begin{aligned} &\nu(\theta) + \frac{2b}{\sqrt{\pi}}\int_0^\theta \frac{\exp\left(-b^2\frac{(\theta-\theta')}{(2+\theta+\theta')}\right)\left(1+\theta'\right)\nu(\theta')}{\sqrt{(2+\theta+\theta')^3(\theta-\theta')}}\mathrm{d}\theta' \\ &+ \frac{\exp\left(-\frac{((1+\theta)b-z)^2}{((1+\theta)^2-1)}\right)}{\sqrt{\pi\left((1+\theta)^2-1\right)}} = 0. \end{aligned} \tag{32}$$

Symbolically,

$$\nu(\theta) + \int_0^\theta \frac{\Phi_b^f(\theta,\theta')\nu(\theta')}{\sqrt{\theta-\theta'}}\mathrm{d}\theta' + \frac{\exp\left(-\frac{((1+\theta)b-z)^2}{((1+\theta)^2-1)}\right)}{\sqrt{\pi\left((1+\theta)^2-1\right)}} = 0,$$

where

$$\Phi_b^f(\theta,\theta') = \frac{2b}{\sqrt{\pi}}\frac{\exp\left(-b^2\frac{(\theta-\theta')}{(2+\theta+\theta')}\right)\left(1+\theta'\right)}{\sqrt{(2+\theta+\theta')^3}}.$$

Accordingly, (31) can be written in the form

$$g(t) = -\frac{\left(e^t b - z\right)\exp\left(-\frac{\left(e^t b - z\right)^2}{\left(e^{2t}-1\right)} + 2t\right)}{\sqrt{\pi\left(e^{2t}-1\right)^3}}$$

$$-\left(e^t b + \frac{e^{2t}}{\sqrt{\pi\left(e^{2t}-1\right)}}\right)\nu(t)$$

$$+\frac{1}{\sqrt{\pi}}e^{2t}\int_0^\theta \frac{\left(\left(1-2b^2\frac{(\theta-\theta')}{(2+\theta+\theta')}\right)\exp\left(-b^2\frac{(\theta-\theta')}{(2+\theta+\theta')}\right)\nu(\theta') - \nu(\theta)\right)\left(1+\theta'\right)}{\sqrt{(2+\theta+\theta')^3(\theta-\theta')^3}}d\theta'.$$

An example of the transformed boundary, for the case of the flat boundary, is given in figure 1.

It is worth noting that the analytical solution is available in two cases:

- $b \equiv 0$: one can be trivially found the solution using the method of images;
- $b(t) = A\,e^{-t} + Be^t$: the boundary transforms to $2B\tau + A + B$. The linear boundary was considered in Section 4.1.

Transformation of the standard OU process to a Wiener process (Backward). Alternatively, we can solve the backward problem:

$$\begin{aligned} G_t(t,T,z) &= -zG_z(t,T,z) + \frac{1}{2}G_{zz}(t,T,z), \\ G(0,T,z) &= 0, \\ G(t,T,b(T-t)) &= 1, \end{aligned} \tag{33}$$

By the same token as before, we introduce

$$\tau = \varpi(t) = \frac{1-e^{-2t}}{2} = e^{-t}\sinh(t),$$

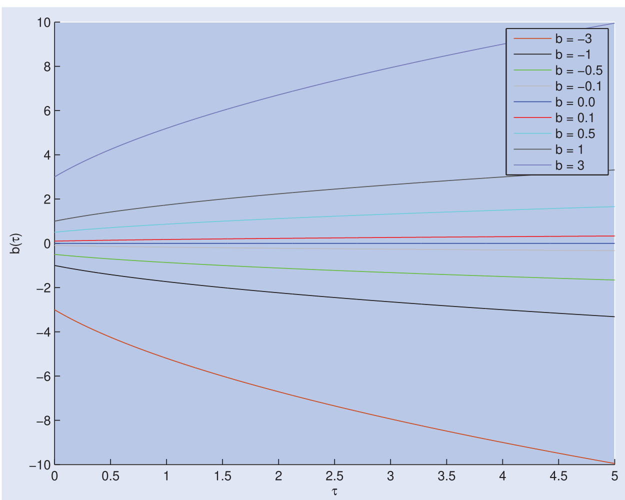

Figure 1. Moving boundary $b(\tau)$ for (4.4).

$$\hat{T} = e^{-T}\sinh(T), \quad 0 \le \tau < \frac{1}{2}, \quad x = e^{-t}z,$$

and rewrite IBVP (33) as follows

$$\begin{aligned} &G_\tau(\tau,x) = \tfrac{1}{2}G_{xx}(\tau,x), \\ &G(0,x) = 0, \\ &G\left(\tau, \beta\left(\hat{T}-\tau\right)\right) = 1, \\ &\beta(\tau) = \sqrt{1-2\tau}\,b(T-t), \end{aligned}$$

It is clear that $0 \le \tau < 1/2$, and $e^{-t} = \sqrt{1-2\tau}$. It is worth noting that for the backward problem the computational domain is compactified in the τ direction. This fact greatly simplifies the numerical evaluation of $G(0,T,z)$, when $T \to \infty$.

Using Theorem 2, the solution is given by (12) and (13).

For the case of flat boundary, $b(t) \equiv b$, we have $\nu(\tau,\hat{T}) \equiv \nu(\tau)$. Accordingly, using Theorem 2,

$$G(\tau,x) = \int_0^\tau \frac{\left(x - b\sqrt{1-2\tau'}\right)\exp\left(-\frac{\left(x-b\sqrt{1-2\tau'}\right)^2}{2(\tau-\tau')}\right)\nu(\tau')}{\sqrt{2\pi(\tau-\tau')^3}}d\tau', \tag{34}$$

Since

$$\frac{b(\tau)-b(\tau')}{\tau-\tau'} = \frac{b\left(\sqrt{1-2\tau}-\sqrt{1-2\tau'}\right)}{\tau-\tau'} = -\frac{2b}{\sqrt{1-2\tau}+\sqrt{1-2\tau'}}.$$

so that (4.4) can be written in the form

$$\nu(\tau) - \sqrt{\frac{2}{\pi}}b\int_0^\tau \times \frac{\exp\left(-b^2\frac{\left(\sqrt{1-2\tau'}-\sqrt{1-2\tau}\right)}{\left(\sqrt{1-2\tau'}+\sqrt{1-2\tau}\right)}\right)\nu(\tau')}{\left(\sqrt{1-2\tau'}+\sqrt{1-2\tau}\right)\sqrt{\tau-\tau'}}d\tau' - 1 = 0. \tag{35}$$

Once (35) is solved, $G(\tau,x)$ and $G(t,z)$ can be calculated by virtue of (34) in a straightforward fashion.

We can rewrite (35) in an alternative way. Let $\vartheta = 1 - \sqrt{1-2\tau}$, $\vartheta' = 1-\sqrt{1-2\tau'}$, $0 \le \vartheta' \le \vartheta < 1$. Then

$$\nu(\vartheta) - \frac{2b}{\sqrt{\pi}}\int_0^\vartheta \times \frac{\exp\left(-b^2\frac{(\vartheta-\vartheta')}{(2-\vartheta-\vartheta')}\right)(1-\vartheta')\nu(\vartheta')}{\sqrt{(2-\vartheta-\vartheta')^3}\sqrt{\vartheta-\vartheta'}}d\vartheta' - 1 = 0. \tag{36}$$

Symbolically,

$$\nu(\vartheta) - \int_0^\vartheta \frac{\Phi_b^b(\vartheta,\vartheta')\nu(\vartheta')}{\sqrt{\vartheta-\vartheta'}}d\vartheta' - 1 = 0,$$

where

$$\Phi_b^b(\vartheta,\vartheta') = \frac{2b}{\sqrt{\pi}}\frac{\exp\left(-b^2\frac{(\vartheta-\vartheta')}{(2-\vartheta-\vartheta')}\right)(1-\vartheta')}{\sqrt{(2-\vartheta-\vartheta')^3}}.$$

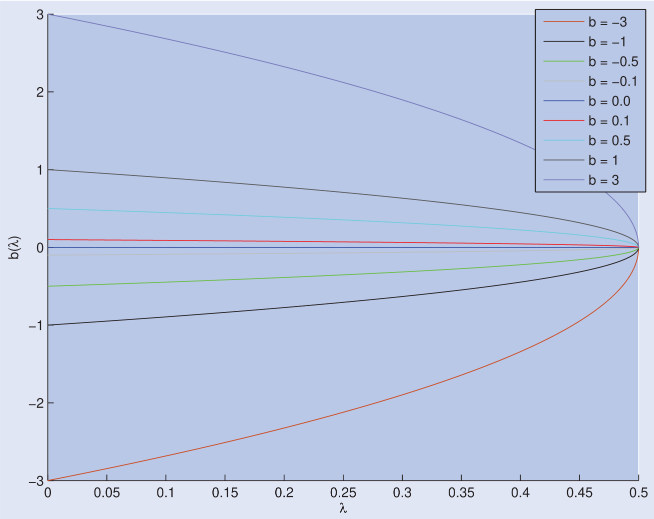

Figure 2. Moving boundary $b(\tau)$ for (4.4).

As one would expect,

$$\Phi_b^b(\theta,\theta') = -i\Phi_{ib}^f\left(-\theta,-\theta'\right).$$

As a result,

$$G(t,z) = 2\int_0^{\vartheta} \frac{\left(z(1-\vartheta) - b\left(1-\vartheta'\right)\right)\exp\left(-\frac{\left(z(1-\vartheta)-b\left(1-\vartheta'\right)\right)^2}{(\vartheta-\vartheta')(2-\vartheta-\vartheta')}\right)\left(1-\vartheta'\right)\nu\left(\vartheta'\right)}{\sqrt{\pi\,(\vartheta-\vartheta')^3\,(2-\vartheta-\vartheta')^3}}\mathrm{d}\vartheta',$$

where $\vartheta = 1 - \mathrm{e}^{-t}$.

We show the moving boundary for different values of b in figure 2.

5. Numerical solution

In this section, we show how to solve the Volterra equations (11) and (13) numerically, and also derive an analytical approximation for small values of t.

5.1. Numerical method

In this section, we briefly discuss two methods to solve the corresponding Volterra equations (11) and (13). We start with a simple trapezoidal method, and then consider a more advanced method based on a quadratic interpolation, which gives a better convergence rate. Both methods can be found in Linz (1985) (Section 8.2 and Section 8.4).

In this section we solve

$$f(t) = g(t) + \int_0^t \frac{K(t,s)}{\sqrt{t-s}} f(s)\,\mathrm{d}s, \tag{37}$$

where $K(t,s)$ is a non-singular part of the kernel.

Both (11) and (13) can be formulated as (37) with an appropriate choice of $K(t,s)$ and $g(t)$.

5.1.1. Simple trapezoidal method. Consider the integral in (37) separately

$$\int_0^t \frac{K(t,s)\nu(s)}{\sqrt{t-s}}\mathrm{d}s = -2\int_0^t K(t,s)\nu(s)\,\mathrm{d}\sqrt{t-s}. \tag{38}$$

We consider a grid $0 = t_0 < t_1 < \cdots < t_N = T$, and denote F_k for the approximated value of $f(t_k)$ and $\Delta_{k,l} = t_k - t_l$. Then, by trapezoidal rule of the Stieltjes integral (38), (37) can be approximated as

$$F_k = g(t_k) + \sum_{i=1}^{k}\left(K(t_k,t_i)F_i + K(t_k,t_{i-1})F_{i-1}\right) \times \left(\sqrt{\Delta_{k,i-1}} - \sqrt{\Delta_{k,i}}\right) = 0.$$

From the last equation, we can express F_k

$$F_k = \left(1 - K(t_k,t_k)\sqrt{\Delta_{k,k-1}}\right)^{-1} \times \left(g(t_k) + K(t_k,0)\left(\sqrt{\Delta_{k,0}} - \sqrt{\Delta_{k,1}}\right) + \sum_{i=1}^{k-1} K(t_k,t_i)\left(\sqrt{\Delta_{k,i-1}} - \sqrt{\Delta_{k,i+1}}\right)F_i\right).$$

Taking $F_0 = g(0)$, we can recursively compute F_m using the previous values $F_0,\ldots,F_{m-1}$.

The approximation error of the integrals is of order $O(\Delta^2)$, where $\Delta = \max_{k,i}\sqrt{\Delta_{k,i-1}} - \sqrt{\Delta_{k,i+1}}$ is the step size. Hence, on the uniform grid $t_i = i\Delta$, the convergence rate is of order $O(h)$.

5.1.2. Block-by-block method based on quadratic interpolation. In order to balance the intuitive appeal of our method with its numerical efficiency we decided to consider block-by-block method based on quadratic interpolation, because it already gives the 3.5 order convergence. However, further improvements based on advanced collocation methods, including Chebyshev and Bernstein polynomials, are possible and can be easily implemented if needed (Maleknejad *et al.* 2007, Kolk and Pedas 2009, Maleknejad *et al.* 2011, Kolk and Pedas 2013).

In this section we follow Linz (1985) (Section 8.4). Using piece-wise quadratic interpolation, Linz (1985) derived

$$\begin{aligned} F_{2m+1} &= g(t_{2m+1}) + (1-\delta_{m0})\sum_{i=0}^{2m} w_{2m+1,i}K(t_{2m+1},t_i)F_i \\ &+ \alpha\left(t_{2m+1},t_{2m},\frac{h}{2}\right)K(t_{2m+1},t_{2m})F_{2m} \\ &+ \beta\left(t_{2m+1},t_{2m},\frac{h}{2}\right)K(t_{2m+1},t_{2m+1/2}) \\ &\times\left(\frac{3}{8}F_{2m} + \frac{3}{4}F_{2m+1} - \frac{1}{8}F_{2m+2}\right) \\ &+ \gamma\left(t_{2m+1},t_{2m},\frac{h}{2}\right)K(t_{2m+1},t_{2m+1})F_{2m+1}, \end{aligned} \tag{39}$$

and

$$
\begin{aligned}
F_{2m+2} = {} & g(t_{2m+2}) + (1-\delta_{m0}) \sum_{i=0}^{2m} w_{2m+2,i} K(t_{2m+2}, t_i) F_i \\
& + \alpha\,(t_{2m+2}, t_{2m}, h)\, K(t_{2m+2}, t_{2m}) F_{2m} \\
& + \beta\,(t_{2m+2}, t_{2m}, h)\, K(t_{2m+2}, t_{2m+1}) F_{2m+1} \\
& + \gamma\,(t_{2m+2}, t_{2m}, h)\, K(t_{2m+2}, t_{2m+2}) F_{2m+2}, \qquad (40)
\end{aligned}
$$

where

$$
\begin{aligned}
w_{n,i} &= (1-\delta_{i,n-1})\alpha(t_n, t_i, h) \\
&\quad + (1-\delta_{i,0})\gamma(t_n, t_i - 2h, h), \quad i \text{ is even}, \\
w_{n,i} &= \beta(t_n, t_i - h, h), \quad i \text{ is odd},
\end{aligned}
$$

and δ_{ij} is a Kronecker delta.

The functions α, β, and γ are defined by

$$
\begin{aligned}
\alpha(x,y,z) &= \frac{z}{2}\int_0^2 \frac{(1-s)(2-s)}{\sqrt{x-y-sz}}\,\mathrm{d}s, \\
\beta(x,y,z) &= z\int_0^2 \frac{s(2-s)}{\sqrt{x-y-sz}}\,\mathrm{d}s, \\
\gamma(x,y,z) &= \frac{z}{2}\int_0^2 \frac{s(s-1)}{\sqrt{x-y-sz}}\,\mathrm{d}s.
\end{aligned}
$$

We note that α, β, and γ depend only on $x-y$, z, and can be written as a function of two variables. Moreover, these integrals can be computed analytically.

Then, one can find $[F_{2m+1}, F_{2m+2}]$ by solving the system of two linear equations (39) and (40). We present the numerical algorithm in Algorithm 1.

Algorithm 1 Block-by-block method based on quadratic interpolation

---

Require: N — number of time steps: $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$
1: $F_0 = f(0)$
2: **for** $m = 0 : N/2$ **do**
3: Compute $w_{2m+1,i}$ for $i = 0, \ldots, 2m$
4: Compute $w_{2m+2,i}$ for $i = 0, \ldots, 2m$
5: Get $[F_{2m+1}, F_{2m+2}]$ by solving (39) and (40)
6: **end for**

Under some assumptions on regularity, the convergence rate of this method is 3.5. In our case, for the backward equation, these assumptions are not satisfied, and we empirically confirm the convergence of order 1.5.

5.2. *Approximation by an Abel integral equation*

Consider the integral equations, which we got for the standard OU process.

For small values of θ, (32) can be approximated by an Abel integral equation of the second kind.

$$
\nu(\theta) + \frac{b}{\sqrt{2\pi}} \int_0^\theta \frac{1}{\sqrt{\theta-\theta'}} \nu(\theta')\,\mathrm{d}\theta' + \frac{\exp\left(-\frac{(b-z)^2}{2\theta}\right)}{\sqrt{2\pi\theta}} = 0. \qquad (41)
$$

The last equation is an Abel equation of the second kind and can be solved analytically using direct and inverse Laplace transforms (Abramowitz and Stegun 1965).

The Laplace transform yields

$$
\bar{\nu}(\Lambda) + b\frac{\bar{\nu}(\Lambda)}{\sqrt{2\Lambda}} + \frac{\mathrm{e}^{-\sqrt{2\Lambda}(z-b)}}{\sqrt{2\Lambda}} = 0.
$$

Then, $\bar{\nu}(\Lambda)$ can be expressed as

$$
\bar{\nu}(\Lambda) = -\frac{\mathrm{e}^{-\sqrt{2\Lambda}(z-b)}}{\sqrt{2\Lambda}+b}
$$

Taking inverse Laplace transform, we get the final expression for $\nu(\theta)$

$$
\nu(\theta) = b\,\mathrm{e}^{b^2/2\theta + b(z-b)} N\left(-\frac{b\theta+z-b}{\sqrt{\theta}}\right) - \frac{\exp\left(-\frac{(b-z)^2}{2\theta}\right)}{\sqrt{2\pi\theta}}, \qquad (42)
$$

where $N(x)$ is the CDF of the standard normal distribution.

Now consider the backward equation. For small values of ϑ, (36) can be approximated by

$$
\nu(\vartheta) - \frac{b}{\sqrt{2\pi}} \int_0^\vartheta \frac{1}{\sqrt{\vartheta-\vartheta'}} \nu(\vartheta')\,\mathrm{d}\vartheta' - 1 = 0. \qquad (43)
$$

Similar to the forward equation, we solve it by taking direct and inverse Laplace transforms. The direct Laplace transform yields

$$
\bar{\nu}(\Lambda) - b\frac{\bar{\nu}(\Lambda)}{\sqrt{2\Lambda}} - \frac{1}{\Lambda} = 0.
$$

Hence,

$$
\bar{\nu}(\Lambda) = \frac{1}{\sqrt{\Lambda}(\sqrt{\Lambda} - b/\sqrt{2})}.
$$

Taking the inverse Laplace transform, we get

$$
\nu(\vartheta) = 2\,\mathrm{e}^{b^2\vartheta/2} N(b\sqrt{\vartheta}). \qquad (44)
$$

Alternatively, one can find an analytical solution using the following results (Polyanin and Manzhirov 1998). The solution of Abel equation

$$
y(t) + \xi \int_0^t \frac{y(s)\,\mathrm{d}s}{\sqrt{t-s}} = f(t).
$$

has the form

$$
y(t) = F(t) + \pi\xi^2 \int_0^t \exp[\pi\xi^2(t-s)] F(s)\,\mathrm{d}s,
$$

where

$$
F(t) = f(t) - \xi \int_0^t \frac{f(s)\,\mathrm{d}s}{\sqrt{t-s}}.
$$

6. Numerical examples

6.1. Numerical tests

Wiener process with linear boundary. Consider $T = 1, z = 2$, and $b(t) = 2t + 1$. In figures 4 and 3 we compare the numerical results with expressions (23) and (24). It is clear that analytical and numerical results are in agreement.

Standard OU process with constant boundary. Consider $T = 2$ and $z = 2$. In figure 5 we examine the density and cumulative density functions for different values of b and compare them with the analytical solution (28) for $b = 0$. In figure 10 we show $G(t, z)$ as a function of both t and z for $b = 1$. In figure 6 we present $\nu(\theta)$ for the forward problem and in figure 7 we present $\nu(\vartheta)$ for the backward problem for various values of b.

To test how good the Abel equation approximates the corresponding Volterra equation, we plot them together for small values of t. In figure 8 we compare the corresponding forward equations and in figure 9 we compare the corresponding backward equations. We can see that up to $t = 0.02$ the solutions are visually indistinguishable.

We also empirically test the convergence rate of our numerical method by implementing it for the forward and backward Abel equations (41) and (43) and comparing them with the analytical solutions (42) and (44). In figure 11a we get the order of convergence 3.2 for the quadratic interpolation method and order 1 for the trapezoidal method for the forward equation. It confirms our estimate in Section 5.1. In figure 11(b) we get the order 1.5 for the quadratic interpolation method and the order 1 for the trapezoidal method for the backward equation. The convergence order of the quadratic interpolation method for the backward equation is smaller than the theoretical estimate because the assumptions on the regularity of the solution are not satisfied. Potentially, a non-uniform grid improves the convergence order.

6.2. Comparison with other methods for the OU process

6.2.1. Description of other methods. We compare our method with other available alternatives.

(1) Leblanc *et al.* (2000) method. We used (3) in Leblanc *et al.* (2000) to compute the hitting density. In our

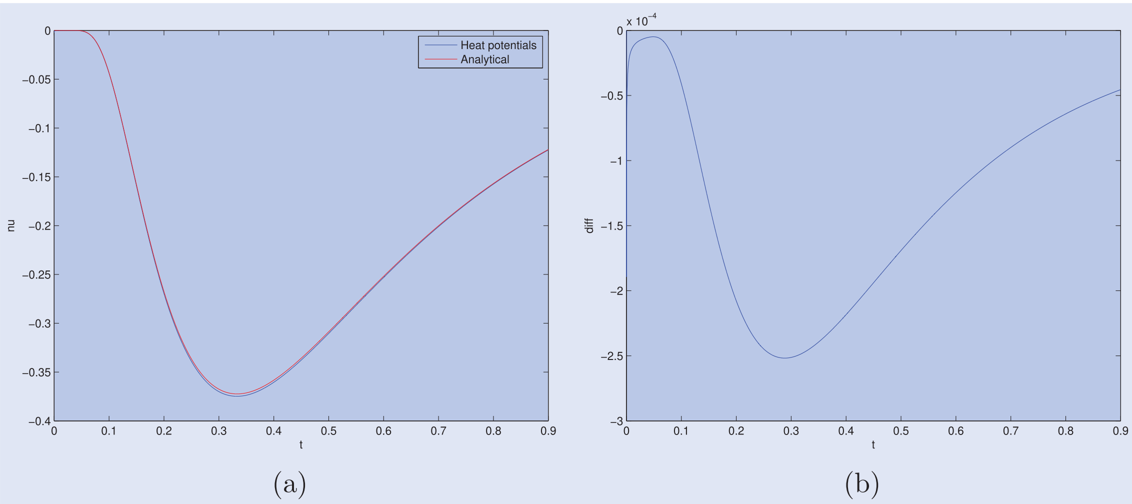

Figure 3. $g(t)$ for the Wiener process and linear boundary (a) on the same plot (b) the difference between heat potentials and analytical solution.

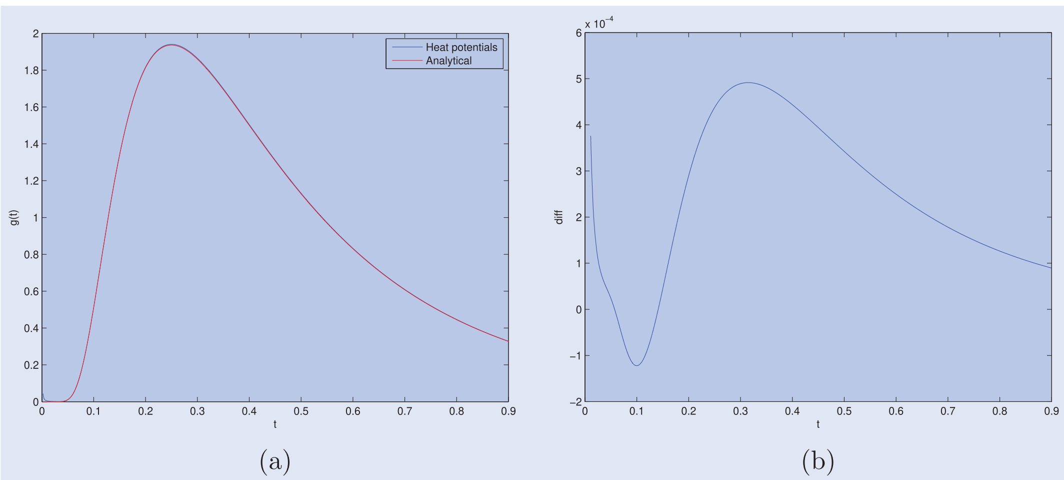

Figure 4. $\nu(t)$ for the Wiener process and linear boundary (a) on the same plot (b) the difference between heat potentials and analytical solution.

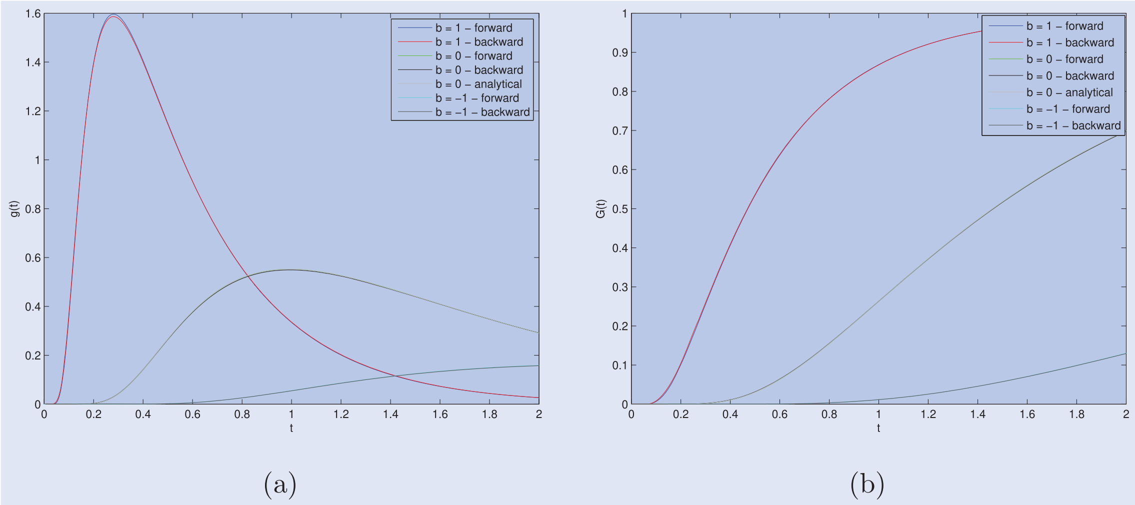

Figure 5. (a) Density function $g(t)$ (b) Cumulative density function $G(t)$.

notation it is

$$g(t) = \frac{(z-b)\exp\left(\frac{z^2-b^2+t-(z-b)^2\coth t}{2}\right)}{\sqrt{2\pi\,(\sinh(t))^3}}$$
$$= \frac{(z-b)\exp\left(b(z-b)-\frac{e^{-t}(z-b)^2}{2\sinh(t)}+\frac{t}{2}\right)}{\sqrt{2\pi\,(\sinh(t))^3}}.$$

Integrating $g(t)$, we get

$$G(t) = 2\,e^{b(z-b)}N\left(-\frac{(z-b)\,e^{-t/2}}{\sqrt{\sinh t}}\right).$$

(2) Alili *et al.* (2005), Linetsky (2004), and Ricciardi and Sato (1988) method. The method is based on the

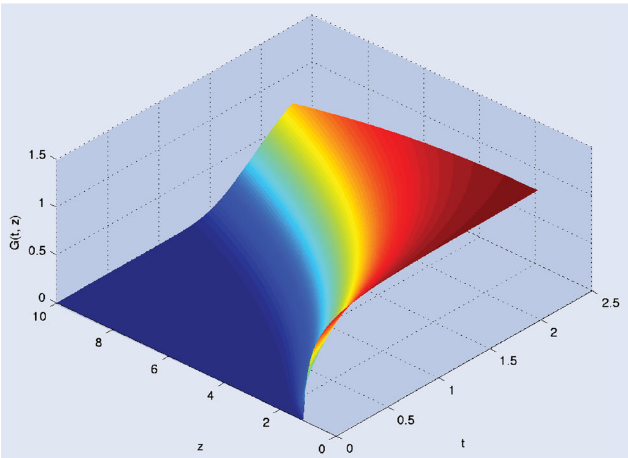

Figure 6. $v(\theta)$, the solution of (32) with $z = 2$ (a) as a function of θ and b (b) as a function of θ for different values of b.

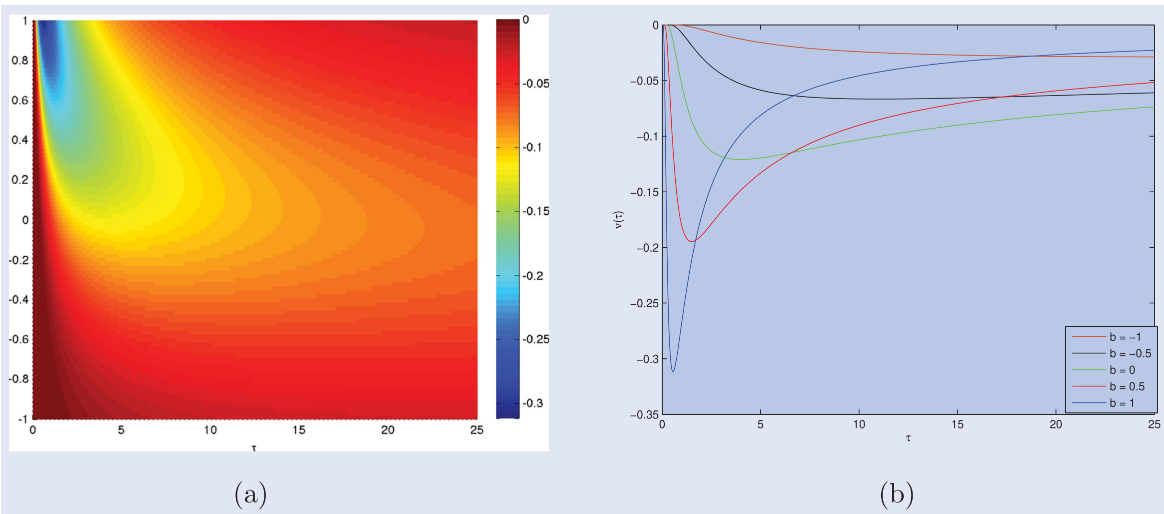

Figure 7. $v(\vartheta)$, the solution of (36) (a) as a function of ϑ and b (b) as a function of ϑ for different values of b.

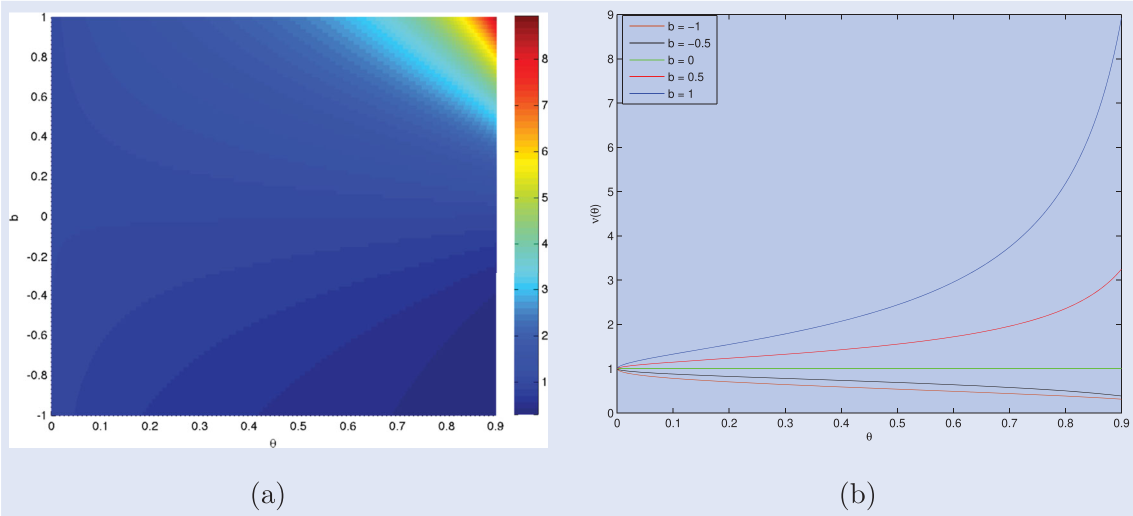

Figure 8. Comparison between the approximation by the Abel equation and the numerical solution of the forward equation $\nu(t)$ in t-coordinates (a) $b = -0.5, z = -0.25$ (b) $b = 0.5, z = 1$.

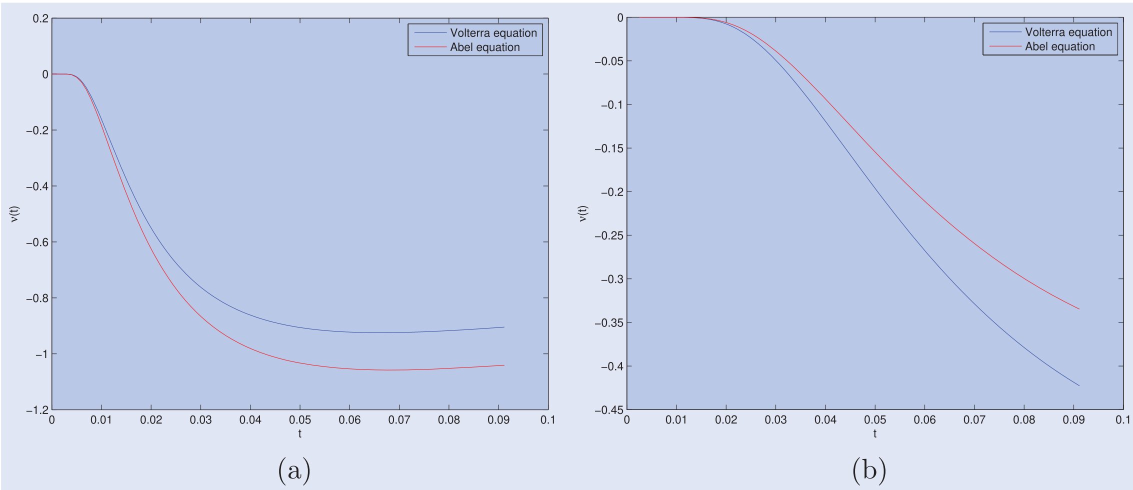

Figure 9. Comparison between the approximation by the Abel equation and the numerical solution of the backward equation $\nu(t)$ in t-coordinates (a) $b = -0.5$ (b) $b = 0.5$.

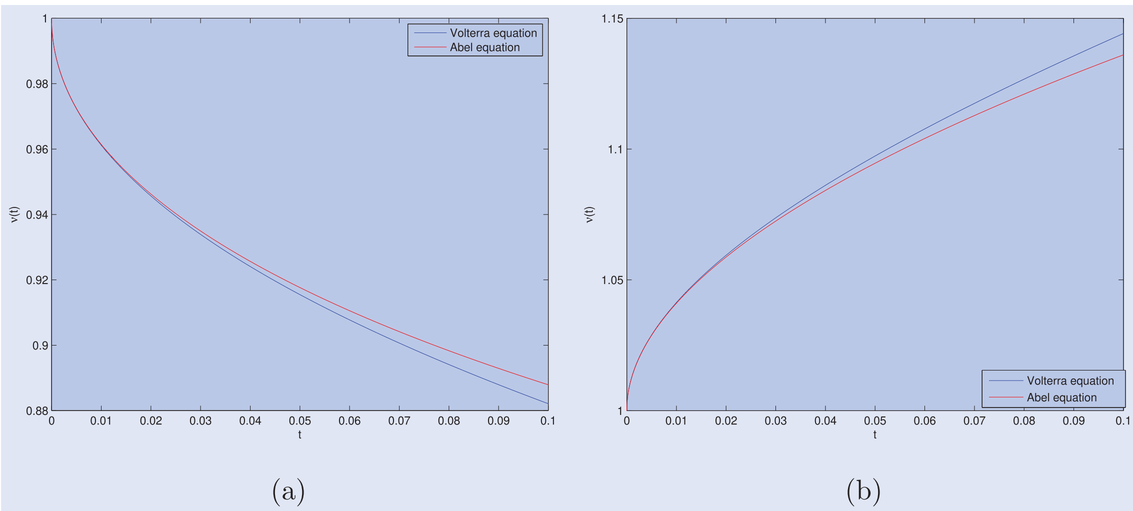

Figure 10. $G(t, z)$ as a function of both t and z for $b = 1$.

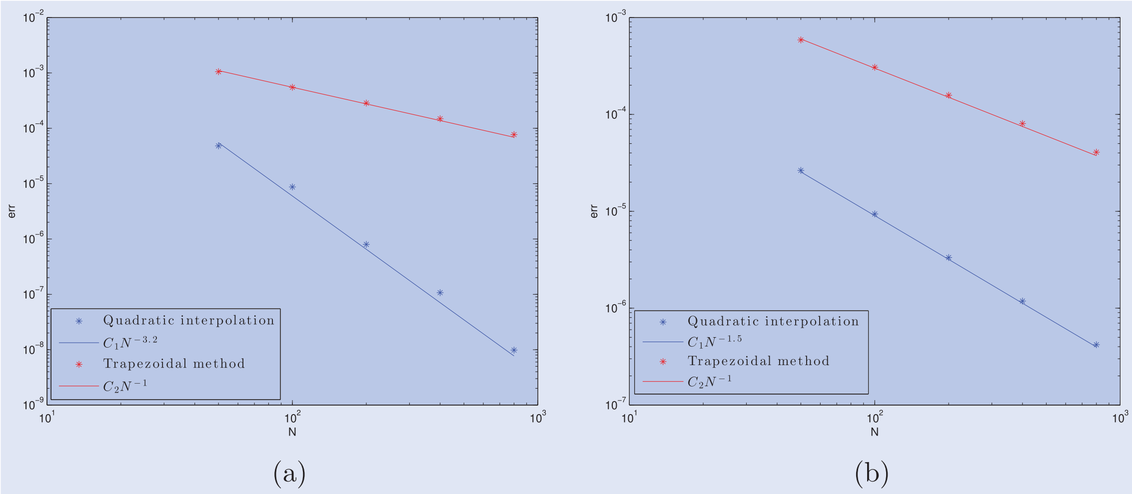

Figure 11. Empirical estimation of the numerical methods from Section 5.1: (a) forward equation, (b) backward equation.

inversion of the Laplace transform of the hitting density. In our notation, the Laplace transform $u(\Lambda)$ has the form

$$u(\Lambda) = \mathrm{e}^{(z^2-b^2/2)} \frac{D_{-\Lambda}(z\sqrt{2})}{D_{-\Lambda}(b\sqrt{2})}, \tag{45}$$

where $D_\nu(x)$ is the parabolic cylinder function.

Alili *et al.* (2005) found a representation of the inverse Laplace transform as a series of parabolic cylinder function and its derivatives

$$g(t) = \mathrm{e}^{(z^2-b^2)/2} \sum_{j=1}^{\infty} \frac{D_{\nu_{j,b\sqrt{2}}}(z\sqrt{2})}{D'_{\nu_{j,b\sqrt{2}}}(b\sqrt{2})} \exp\left(\nu_{j,b\sqrt{2}}\right),$$

where $D'_\nu(x) = (\partial D/\partial \nu)(x)$ and $\nu_{j,b}$ the ordered sequence of positive zeros of $\nu \to D_\nu(x)$.

However, we used the Gaver–Stehfest algorithm (Abate *et al.* 2000) for the inversion, as it gives more stable and robust results. Moreover, the representation from Alili *et al.* (2005) works only for $t > t_0$ for some t_0, while the Gaver–Stehfest algorithm works for all positive t.

Using this method, the density can be expressed as

$$g(t) \approx \frac{\ln 2}{t} \sum_{k=1}^{2m} \omega_k u\left(\frac{k \ln 2}{t}\right),$$

where

$$\omega_k = (-1)^{m+k} \sum_{j=\lfloor (k+1)/2 \rfloor}^{\min(k,m)} \frac{j^{m+1}}{m} C^j_m C^j_{2j} C^j_{k-j}.$$

As we can see, the method only requires the values of u on the positive real semi-axis, and from (45) one can observe that $u(\Lambda)$ is non-singular on $\mathbb{R}_+$. As an example, in figure 12, we plot $u(\Lambda)$ for $\Lambda > 0$ with $z = 2$ and various values of b.

As an example of using the Gaver–Stehfest algorithm in finance, one can refer to Lipton and Sepp (2011b), where the authors used it for the calibration of a local volatility surface.

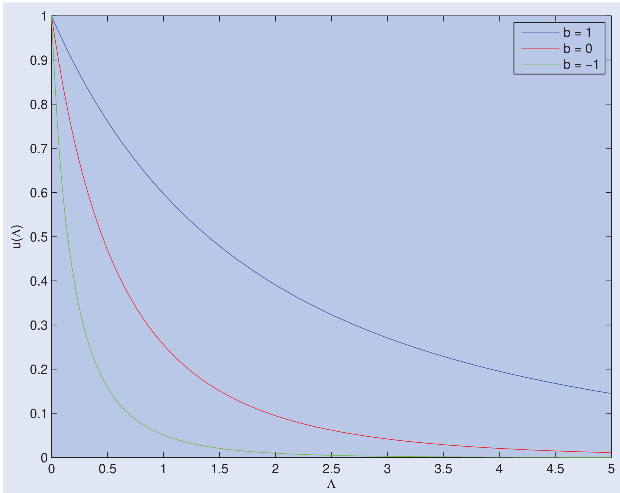

Figure 12. Laplace transform $u(\Lambda)$ for $z = 2$.

(3) Crank–Nicolson method. We solved (33) using Crank–Nicolson numerical scheme (Duffy 2013). We should note that other more accurate numerical methods are available (embedded Runge–Kutta methods, Gear's methods, etc.). However, for our illustrative purposes, we decided to compare our results with a simple Crank–Nicolson scheme.

6.2.2. Comparison. We start with a comparison with Leblanc *et al.* (2000) method to show that it gives wrong results. In figure 13 we compare two methods for different parameters. We also give the analytical solution, when it is available for $b = 0$. We can observe that only for $b = 0$ the Leblanc *et al.* (2000) method gives correct results, while for other parameters it totally differs from our method. Moreover in figure 13(c) we can see that it gives $G(t) > 1$.

A closed-form solution is only available when $b = 0$ and is given by (28). We take other parameters as before $T = 2$ and $z = 2$. We compare our method (forward and backward),

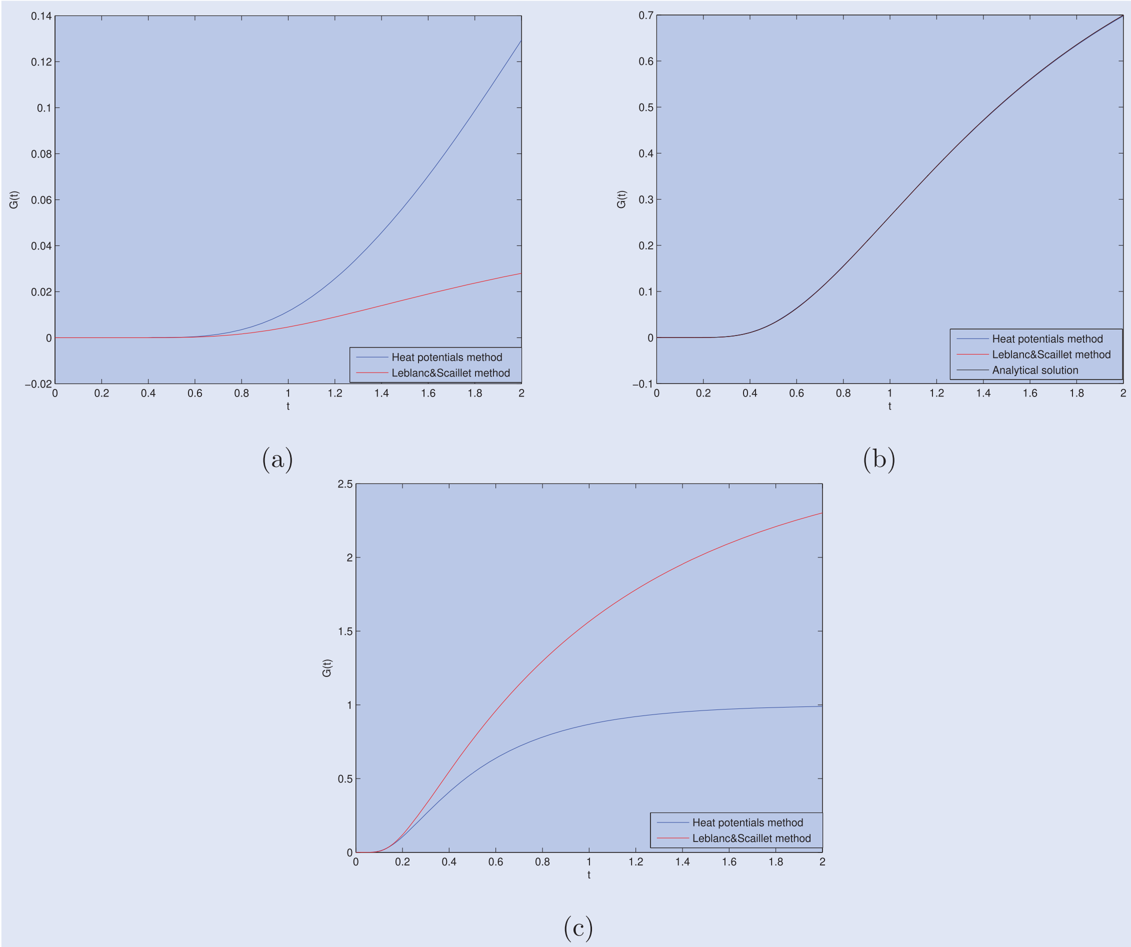

Figure 13. Comparison with Leblanc *et al.* (2000) method: (a) $z = 2, b = -1$, (b) $z = 2$, $b = 0$, (c) $z = 2$, $b = -1$.

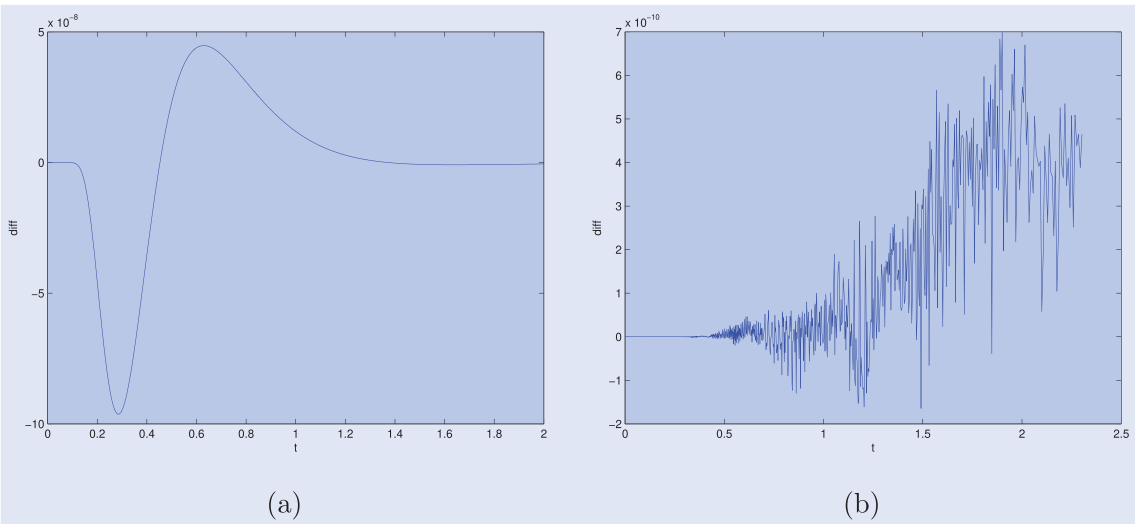

Figure 14. The difference between numerical and analytical solutions (a) Crank–Nicolson method (b) Laplace transform method.

the Crank–Nicolson method, and the method based on the inversion of the Laplace transform in figures 15 and 14. We note that our method has an advantage for this case because Volterra equations (30) and (36) become trivial for $b = 0$. Nevertheless, the comparison is still very useful.

We take $N = 500$ for both forward and backward methods, $h = k = 0.005$ for the Crank–Nicolson method, and $m = 24$ for the Gaver–Stehfest algorithm. We used `mpmath` library in `Python` (Johansson 2013) for the implementation of the Gaver–Stehfest algorithm and parabolic cylinder functions with arbitrary precision arithmetics. In our computations, we used 100 digits precision.

From these graphs, we observe that the methods developed in this paper and the method based on the inversion of the

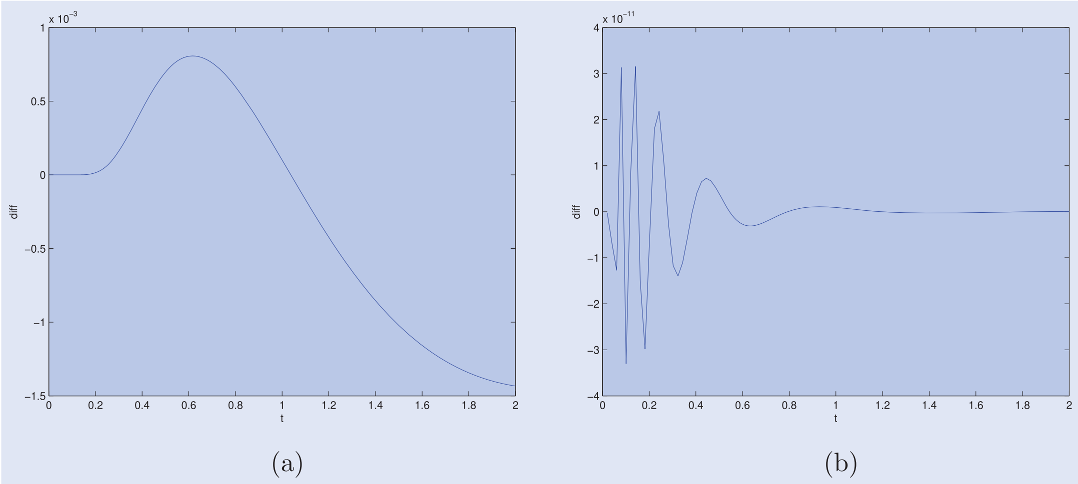

Figure 15. The difference between numerical and analytical solutions for $b = 0$ (a) Forward method (b) Backward method.

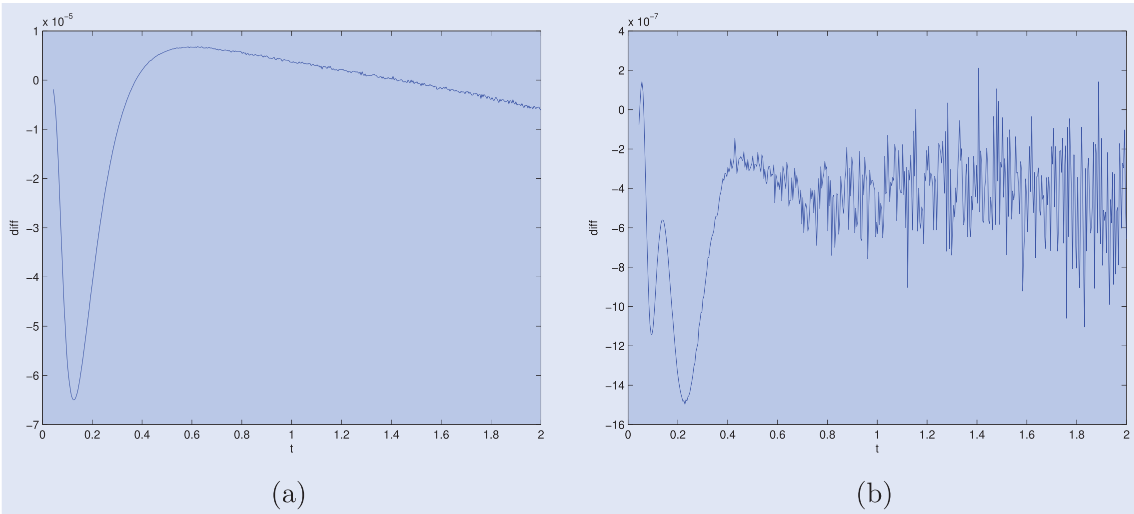

Figure 16. The difference between our method and the Laplace transform method (a) $N = 100$ (b) $N = 10000$.

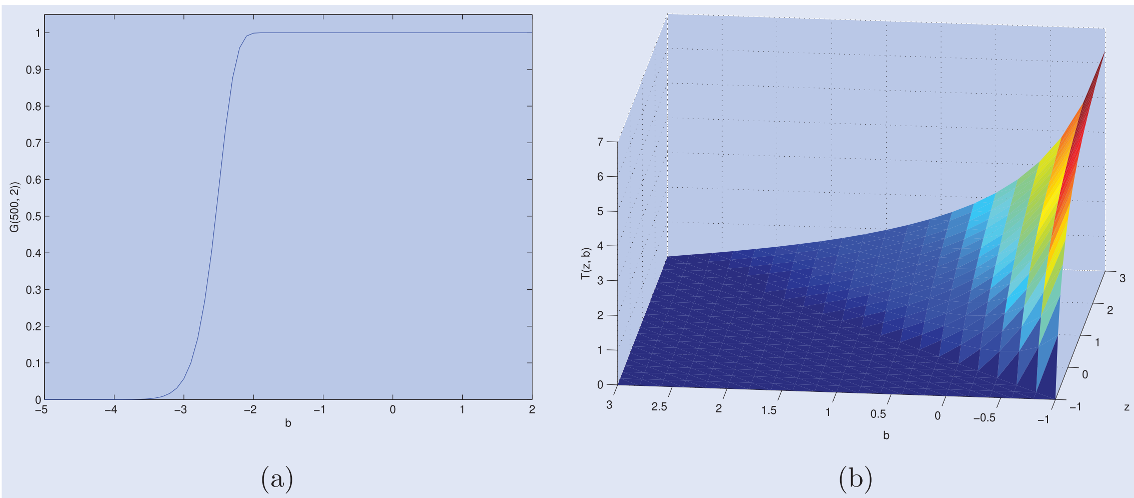

Figure 17. (a) $G(T, z)$ as a function of b with $T = 500$ and $z = 2$ (b) $T(z, b) = \mathbb{E}[s]$, where s is the hitting time.

Laplace transform (Alili *et al.* 2005) significantly outperform the Crank–Nicolson scheme.

In order to compare the solutions for non-trivial parameters, we take the Laplace transform method with a maximum possible precision, and our method for $N = 100$ and $N = 10000$. The first example shows how our results can be comparable to the Laplace transform method for a relatively small value of N when the computations can be done very fast; the second example demonstrates that our method can obtain a very good precision for a large N. The results are presented in figure 16.

6.3. Asymptotic behavior when $t \to \infty$ for standard OU process

It is clear that for $b = 0$, and hence all $b \geq 0$, $G(t, z) \to 1$, when $t \to \infty$. But it is unclear what happens for a negative b. We empirically investigate it by plotting $G(t, z)$ as a function of b for a fixed large value of T and fixed z. In figure 17(a) we take $z = 2$ and $T = 500$ and compute $G(T, z)$ for $b \in [-5, 2]$. We can see that it still remains close to 1 up to $b = -2$, and then rapidly approaches zero. In further research, we want to explore the asymptotic behavior of $G(t)$ in more details. In figure 17(b) we show the expected hitting time; this quantity is very important for the design of mean-reverting trading strategies since it allows a trader to decide when to go in and out of the trade.

7. Financial applications

The hitting density of a Wiener and OU processes to a curvilinear boundary has many applications in applied mathematics, especially in mathematical finance. The classical problem of pricing a barrier option in presence of term structure volatility. As it is shown in Lipton (2001) (pp. 484–486), the problem can be easily transformed to the problem with constant coefficients, but with the curvilinear barrier, which belongs to the class of problems we consider in this paper. In this paper, we argue that this problem can easily be solved using the method heat potentials instead of solving the problem numerically using the finite-difference method.

Pricing methods for path-dependent options on yields were proposed in Leblanc and Scaillet (1998), where the authors reduced it to the hitting problem of an Ornstein–Uhlenbeck process.

Another important problem is the design of mean-reverting trading strategies and pairs trading, which was considered in Avellaneda and Lee (2010) and De Prado (2018). Credit risk modeling in a structural framework is another topic, which heavily uses the first passage density of various stochastic processes. We refer to Black and Cox (1976), Hyer *et al.* (1999), Hull and White (2001), Avellaneda and Zhu (2001), Collin-Dufresne and Goldstein (2001), Coculescu *et al.* (2008), Yi (2010), Cheridito and Xu (2015), Lipton and Sepp (2011a), Lipton and Savescu (2014) just to mention a few. The detailed review can be found in Lipton and Sepp (2011a).

It has also applications in quantitative biology (Smith 1991), where the hitting time is used for modeling the time between rings of a nerve cell, and numerous other fields.

7.1. Barrier option pricing

For simplicity, consider a single one-touch digital option. The option pays 1 dollar immediately when the stock price hits the barrier B.

Assume the stock price is driven by

$$\frac{\mathrm{d}S_t}{S_t} = r(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t,$$
$$S_0 = z.$$

Hence, the value of this option can be given by

$$V = \int_0^{\mathrm{T}} \mathrm{e}^{-\int_0^t r(s)\,\mathrm{d}s} g(t, z)\,\mathrm{d}t,$$

where $g(t, z)$ is given in (25) with $\delta(t) = r(t)$.

7.2. Pairs trading

As we mentioned, the hitting density of an OU process might be very useful for constructing mean-reverting trading strategies. Consider two stocks P and Q. We can model them together as

$$\frac{\mathrm{d}P_t}{P_t} = \alpha\,\mathrm{d}t + \beta\frac{\mathrm{d}Q_t}{Q_t} + \mathrm{d}X_t,$$

where X_t is called the cointegration residual.

Usually, the value of α is quite small. It means that one can construct the portfolio by buying 1 dollar of P and selling β dollars of Q. This portfolio will be driven by X_t, which in many cases is observed to be an Ornstein–Uhlenbeck process. As a result a trader might use the following strategy:

(1) go into the trade when $X = X_{\text{in}} > \theta$, say;
(2) when the portfolio reaches a certain level $X = X_{\text{out}} < \theta$, go out of the trade.

By using the first hitting time density, the trader might forecast the exit time from the trade given a level of profitability or the level of profitability given the expected time of closing her position by using, for example, figure 17(b). Optimization of the corresponding strategy is clearly possible as well. The power of our method is particularly clear when the residual is driven by the OU process with time-dependent parameters.

7.3. Credit risk modeling

Consider a bank, whose assets $A(t)$ are driven by geometric Brownian motion and liabilities $L(t)$ are deterministic. The default is defined as

$$\tau = \inf\{t \geq 0 \mid A(t) \leq L(t)\}.$$

As a result, the problem of computation of default probability can be reduced to the hitting problem of a Wiener process to a boundary $b(t)$, which can be easily computed by the methods described in this paper.

Moreover, the method allows to solve efficiently the inverse (calibration) problem, which is more difficult. Assume we are

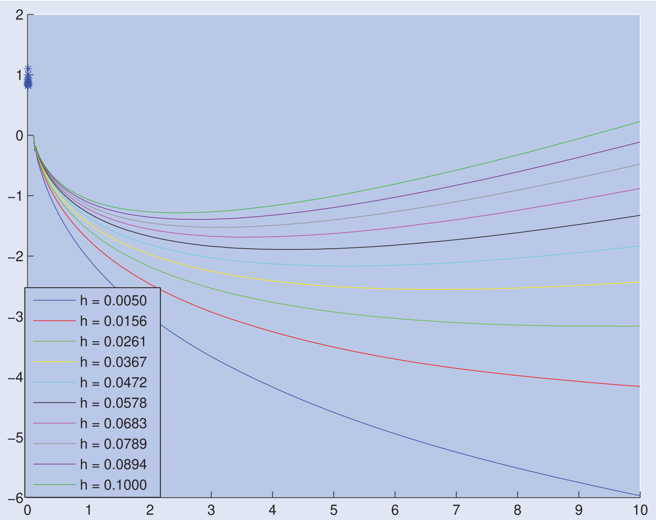

Figure 18. Default boundary $b(t)$ computed using Algorithm 2 for default probability $\pi(t) = 1 - \mathrm{e}^{-ht}$ for different h; corresponding values of z are denoted as starts.

given the default probability curve $\pi(t)$. Then, we need to find the default boundary $b(t)$ to match $\pi(t)$:

$$\pi(t) = \pi^*(t), \tag{46}$$

where

$$\pi^*(t) = \int_0^t g(t', z)\,\mathrm{d}t'.$$

We approximate the default boundary by a piece-wise linear function

$$b(t_i) = \sum_{j=1}^{i} \gamma_j (t_j - t_{j-1}). \tag{47}$$

The calibration can be done using a simple iterative algorithm, which on each step finds γ_i to match the default boundary. The details are given in Algorithm 2 and a numerical example in figure 18. Detailed results will be given elsewhere.

Algorithm 2 Numerical method for the inverse problem

Determine $b(t_1)$ by matching the short-term default probability.
for $i = 2 : n$ **do**
    Compute (47) as a function of γ_i.
    Find $\nu(t)$ as a solution of (11).
    Solve (46) with respect to unknown γ_i.
    Compute $b(t_i)$ using γ_i found before.
end for

8. Conclusion

We developed a semi-analytical approach to finding the first hitting time density of a general diffusion process reducible to a standard Wiener process. We transformed our problem to an initial boundary value problem. By using the method of heat potentials, we derived the corresponding expression for the density via a weight function, which satisfies a Volterra equation of the second kind. To solve the equation, we developed a numerical recursive scheme. We showed order of convergence 3 for the forward scheme and order 1.5 for the backward scheme, and confirmed these numerically.

We demonstrated several representative examples, such as a Wiener process, a Geometric Brownian motion, and both time-dependent and time-independent OU processes.

For the case of a standard OU process, we compared our method to several other known methods in the literature. We showed that our method significantly outperforms the Crank–Nicolson scheme and is at least as good as the Laplace transform method while being markedly more stable. In addition, our method is equally applicable to the case of a time-dependent OU process, whilst the methods based on the Laplace transform are able to deal only with the constant coefficients. We leave comparison with other advanced numerical methods for future research, but note that our method is much easier to implement than these.

It is worth mentioning that the method of heat potentials can be extended to two dimensional hitting problems which are reducible to equivalent hitting problems for a standard two dimensional Wiener process. Of course, the method results in the Volterra–Fredholm integral equation, which requires careful treatment, see Landis (1997), but is still easier to deal with than conventional methods. The corresponding results will be reported elsewhere. Studying multidimensional problems is harder due to their very nature and is left to the future.

As we pointed out, the problem has many practical applications both in physics and finance, such as barrier options pricing, construction of mean-reverting trading strategies, and credit risk modeling. Other applications will be discussed elsewhere.

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Disclosure statement

No potential conflict of interest was reported by the authors.

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Proof of Theorem 1

To be concrete, let us assume that $z > b(0)$. Then

$$g(t) = \frac{1}{2} p_x(t, b(t)) = \frac{1}{2} \lim_{x \to b(t)} p_x(t, x) =$$

$$= -\frac{(z - b(t)) \exp\left(-\frac{(z-b(t))^2}{2t}\right)}{2\sqrt{2\pi t^3}} + \frac{1}{2} \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon}$$

$$\times \int_0^t \frac{\left(\varepsilon + \Psi\left(t,t'\right)\right)\exp\left(-\frac{(\varepsilon+\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\nu\left(t'\right)\mathrm{d}t'$$

$$= -\frac{\left(z - b\left(t\right)\right)\exp\left(-\frac{(z-b(t))^2}{2t}\right)}{2\sqrt{2\pi t^3}}$$

$$+\frac{1}{2}\lim_{\varepsilon\to 0}\int_0^t\left(1 - \frac{\left(\varepsilon+\Psi\left(t,t'\right)\right)^2}{\left(t-t'\right)}\right)$$

$$\times\frac{\exp\left(-\frac{(\varepsilon+\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\nu\left(t'\right)\mathrm{d}t'.$$

We compute the corresponding limit by splitting it into four parts:

$$\lim_{\varepsilon\to 0}\int_0^t\left(1 - \frac{\left(\varepsilon+\Psi\left(t,t'\right)\right)^2}{\left(t-t'\right)}\right)\frac{\exp\left(-\frac{(\varepsilon+\Psi(t,t'))^2}{2(t-t')}\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\nu\left(t'\right)\mathrm{d}t'$$

$$= \lim_{\varepsilon\to 0}\int_0^t\left(1 - \frac{\varepsilon^2}{t-t'} - 2\frac{\varepsilon\Psi\left(t,t'\right)}{t-t'} - \frac{\Psi\left(t,t'\right)^2}{t-t'}\right)$$

$$\times\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon\Psi(t,t')}{t-t'}\right)\Xi(t,t')}{\sqrt{2\pi\left(t-t'\right)^3}}\nu\left(t'\right)\mathrm{d}t'$$

$$= \mathbb{L}_1 + \mathbb{L}_2 - 2\mathbb{L}_3 - \mathbb{L}_4,$$

where

$$\mathbb{L}_1 = \nu(t)\lim_{\varepsilon\to 0}\int_0^t\left(1 - \frac{\varepsilon^2}{t-t'}\right)\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon\Psi(t,t')}{t-t'}\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t',$$

$$\mathbb{L}_2 = \lim_{\varepsilon\to 0}\int_0^t\left(1 - \frac{\varepsilon^2}{t-t'}\right)$$

$$\times\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon\Psi(t,t')}{t-t'}\right)\left(\Xi(t,t')\nu(t') - \nu(t)\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t',$$

$$\mathbb{L}_3 = \lim_{\varepsilon\to 0}\int_0^t\frac{\varepsilon\Psi\left(t,t'\right)}{t-t'}\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon\Psi(t,t')}{t-t'}\right)\Xi(t,t')}{\sqrt{2\pi\left(t-t'\right)^3}}\nu\left(t'\right)\mathrm{d}t'$$

$$\mathbb{L}_4 = \lim_{\varepsilon\to 0}\int_0^t\frac{\Psi\left(t,t'\right)^2}{t-t'}\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon\Psi(t,t')}{t-t'}\right)\Xi(t,t')}{\sqrt{2\pi\left(t-t'\right)^3}}\nu\left(t'\right)\mathrm{d}t',$$

and

$$\Psi\left(t,t'\right) = b\left(t\right) - b\left(t'\right),$$

$$\Xi(t,t') = \exp\left(-\frac{\left(b\left(t\right) - b\left(t'\right)\right)^2}{2\left(t-t'\right)}.\right)$$

We have

$$\mathbb{L}_1 = \nu(t)\lim_{\varepsilon\to 0}\int_0^t\left(1 - \frac{\varepsilon^2}{\left(t-t'\right)}\right)\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon(b(t)-b(t'))}{t-t'}\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t'$$

$$= \nu\left(t\right)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{t/\varepsilon^2}\left(1 - \frac{1}{u}\right)\frac{\exp\left(-\frac{1}{2u}\right)}{\sqrt{2\pi u^3}}\mathrm{d}u$$

$$= 2\nu\left(t\right)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\varepsilon/\sqrt{t}}^{\infty}\left(1 - \nu^2\right)\frac{\exp\left(-\frac{\nu^2}{2}\right)}{\sqrt{2\pi}}\mathrm{d}\nu$$

$$= -2\nu\left(t\right)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon/\sqrt{t}}\left(1 - \nu^2\right)\frac{\exp\left(-\frac{\nu^2}{2}\right)}{\sqrt{2\pi}}\mathrm{d}\nu$$

$$= -\frac{2}{\sqrt{2\pi t}}\nu\left(t\right),$$

where $t - t' = \varepsilon^2 u$, $u = 1/\nu^2$ and we have used the fact that

$$\int_0^{\infty}\left(1 - \nu^2\right)\frac{\exp\left(-\frac{\nu^2}{2}\right)}{\sqrt{2\pi}}\mathrm{d}\nu = 0.$$

Further,

$$\mathbb{L}_2 = \lim_{\varepsilon\to 0}\int_0^t\left(1 - \frac{\varepsilon^2}{t-t'}\right)$$

$$\times\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon(b(t)-b(t'))}{t-t'}\right)\left(\Xi(t,t')\nu(t') - \nu(t)\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t'$$

$$= \lim_{\varepsilon\to 0}\int_0^t\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon(b(t)-b(t'))}{t-t'}\right)\left(\Xi(t,t')\nu(t') - \nu(t)\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t'$$

$$= \int_0^t\frac{\left(\Xi(t,t')\nu(t') - \nu(t)\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t'$$

where we have dropped the higher order ε^2 term in the integral in the second line. $\mathbb{L}_3$ is computed as in Tikhonov and Samarskii (1963, pp. 530–535)

$$\mathbb{L}_3 = \lim_{\varepsilon\to 0}\int_0^t\frac{\varepsilon(b(t) - b(t'))}{t-t'}$$

$$\times\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon(b(t)-b(t'))}{t-t'}\right)\Xi(t,t')}{\sqrt{2\pi\left(t-t'\right)^3}}\nu\left(t'\right)\mathrm{d}t'$$

$$= b'(t)\Xi\left(t,t\right)\nu\left(t\right),$$

and

$$\mathbb{L}_4 = \lim_{\varepsilon\to 0}\int_0^t\frac{(b(t) - b(t'))^2}{t-t'}$$

$$\times\frac{\exp\left(-\frac{\varepsilon^2}{2(t-t')} + \frac{\varepsilon(b(t)-b(t'))}{t-t'}\right)\Xi(t,t')}{\sqrt{2\pi\left(t-t'\right)^3}}\nu\left(t'\right)\mathrm{d}t'$$

$$= \int_0^t\left(\frac{b(t) - b(t')}{t-t'}\right)^2\frac{\Xi\left(t,t'\right)}{\sqrt{2\pi\left(t-t'\right)}}\nu\left(t'\right)\mathrm{d}t'.$$

Combining all terms, we finally have

$$g\left(t,z\right) = \frac{1}{2}\int_0^t\frac{\left(1 - \frac{\Psi(t,t')^2}{(t-t')}\right)\Xi\left(t,t'\right)\nu\left(t'\right) - \nu\left(t\right)}{\sqrt{2\pi\left(t-t'\right)^3}}\mathrm{d}t'$$

$$-\frac{1}{\sqrt{2\pi t}}\nu\left(t\right)$$

$$+\left(-b'\left(t\right)\nu\left(t\right) + \frac{\left(z - b\left(t\right)\right)\exp\left(-\frac{(z-b(t))^2}{2t}\right)}{2\sqrt{2\pi t^3}}\right).$$