A closed-form solution for optimal mean-reverting trading strategies

All market makers are confronted with the problem of defining profit-taking and stop-out levels. More generally, all execution traders holding a particular position for a client must determine at what levels an order needs to be fulfilled. Those optimal levels can be ascertained by maximising the trader's Sharpe ratio in the context of Ornstein-Uhlenbeck processes via Monte Carlo experiments. In this article, Alex Lipton and Marcos Lopez de Prado develop an analytical framework and derive those optimal levels by using the method of heat potentials

Mean-reverting trading strategies in various contexts have been studied for decades. For instance, Elliott *et al* (2005) explained how mean-reverting processes might be used in pairs trading and developed several methods for parameter estimation. Avellaneda & Lee (2010) used mean-reverting processes for pairs trading and modelled the hitting time to find the exit rule of the trade. Bertram (2010) developed some analytic formulas for statistical arbitrage trading where the security price follows an Ornstein-Uhlenbeck (OU) process. Lindberg (2014) models the spread between two assets as an OU process and studies the optimal liquidation strategy for an investor who wants to optimise profit over opportunity cost. Lopez De Prado (2018, chapter 13) considered trading rules for discrete-time mean-reverting trading strategies and found optimal trading rules using Monte Carlo simulations. We emphasise that most, if not all, of the analytical results derived by the above authors are asymptotic and valid for perpetual trading strategies only (see, for example, Bertram 2010; Lindberg 2014). While interesting from a theoretical standpoint, they are challenging to use in practice. In contrast, our approach deals with finite maturity trading strategies and, because of that, has immediate applications.

When prices reflect all available information, they oscillate around an equilibrium level. This oscillation is the result of the temporary market impact caused by waves of buyers and sellers. The resulting price behaviour can be approximated through an OU process. The parameters of the process might be estimated using historical data.

Market makers provide liquidity in an attempt to monetise this oscillation. They enter a long position when a security is priced below its estimated equilibrium level, and they enter a short position when a security is priced above its estimated equilibrium level. They hold that position until one of three outcomes occurs: (a) they achieve a targeted profit, (b) they experience a maximum tolerated loss or (c) the position is held beyond a maximum tolerated horizon.

All traders are confronted with the problem of defining profit-taking and stop-out levels. More generally, all execution traders holding a particular position for a client must determine at what levels an order needs to be fulfilled. Lopez De Prado (2018, chapter 13) explains how to determine those optimal levels in the sense of maximising the trader's Sharpe ratio (SR) in the context of OU processes via Monte Carlo experiments. Although Lopez De Prado (2018, p. 192) conjectured the existence of an analytical solution to this problem, he identified it as an open problem. In this article, we solve the critical problem of finding optimal trading rules analytically by using the method of heat potentials. These optimal profit-taking/stop-loss trading rules for mean-reverting trading strategies provide the algorithm we must follow to exit a position. To put it differently, we find the optimal exit corridor to maximise the SR of the strategy.

The method of heat potential is a highly powerful and versatile approach, popular in mathematical physics (see, for example, Tikhonov & Samarskii 1963). It has been successfully used in numerous important fields, such as thermal engineering, nuclear engineering and material science. However, it is not particularly popular in mathematical finance, even though the first important use case was given by Lipton almost 20 years ago. Specifically, Lipton (2001, section 12.2.3, pp. 462–467) considered pricing barrier options with curvilinear barriers. More recently, Lipton & Kaushansky (2020a,b) described several important financial applications of the method.

The SR is defined as the ratio between the expected returns of an execution algorithm and the standard deviation of the same returns. The returns are computed as the logarithmic ratio between the exit and entry prices multiplied by the sign of the order side (+1 for a sell order, −1 for a buy order). We choose SR as an objective function for two reasons: (a) the SR is the most popular criterion for investment efficiency (Bailey & Lopez De Prado 2013); and (b) the SR can be understood as a t-value of the estimated gains and modelled accordingly for inferential purposes. The distributional properties of the SR are well-known, and this statistic can be deflated when the assumption of normality is violated (Bailey & Lopez De Prado 2014). It is worth noting that some practitioners measure the SR using a pathwise approach. In the latter approach, the SR is a random quantity associated with trading over a specific period. The expected SR can be optimised mathematically.

Having an analytical estimation of the optimal profit-taking and stop-out levels allows traders to deploy tactical execution algorithms, with maximal expected SR. Rather than deriving an 'all-weather' execution algorithm, which supposedly works under every market regime, traders can use our analytical solution to deploy an algorithm that maximises the SR under the prevailing market regime.

Definitions of variables

Suppose an investment strategy S invests in $i = 1, \dots, I$ opportunities or bets. At each opportunity i, S takes a position of m_i units of security X, where $m_i \in (-\infty, \infty)$. The transaction that entered such an opportunity was priced at a value of $m_i P_{i,0}$, where $P_{i,0}$ is the average price per unit at which the m_i securities were transacted. As other market participants transact security X, we can mark-to-market (MtM) the value of that opportunity i after t observed transactions as $m_i P_{i,t}$. This represents the value of opportunity i if it were liquidated at the price observed in the market after t transactions. Accordingly, we can compute the MtM profit/loss of opportunity i after t transactions as:

$$\pi_{i,t} = m_i(P_{i,t} - P_{i,0})$$

A standard trading rule provides the logic for exiting opportunity i at $t = T_i$. This occurs as soon as one of two conditions is verified:

- $\pi_{i,T_i} \geqslant \overline{\pi}$, where $\overline{\pi} > 0$ is the profit-taking threshold;
- $\pi_{i,T_i} \leqslant \underline{\pi}$, where $\underline{\pi} < 0$ is the stop-loss threshold.

Because $\underline{\pi} < \overline{\pi}$, only one of the two exit conditions can trigger the exit from opportunity i. Assuming opportunity i can be exited at T_i, its final profit/loss is π_{i,T_i}. At the onset of each opportunity, the goal is to realise an expected profit:

$$E_0[\pi_{i,T_i}] = m_i(E_0[P_{i,T_i}] - P_{i,0})$$

where $E_0[P_{i,T_i}]$ is the forecasted price and $P_{i,0}$ is the entry level of opportunity i.

Parameter estimation

Consider the discrete OU process on a price series $\{P_{i,t}\}$:

$$P_{i,t} - E_0[P_{i,T_i}] = \kappa(E_0[P_{i,T_i}] - P_{i,t-1}) + \sigma\varepsilon_{i,t}$$

such that the random shocks are independently and identically distributed $\varepsilon_{i,t} \sim \mathcal{N}(0,1)$. The seed value for this process is $P_{i,0}$, the level targeted by opportunity i is $E_0[P_{i,T_i}]$, and κ determines the speed at which $P_{i,0}$ converges towards $E_0[P_{i,T_i}]$.

We estimate the input parameters $\{\kappa, \sigma\}$ by stacking the opportunities as:

$$\begin{aligned} X &= (E_0[P_{0,T_0}] - P_{0,0}, E_0[P_{0,T_0}] - P_{0,1}, \dots, E_0[P_{0,T_0}] - P_{0,T-1}, \\ &\qquad \dots, E_0[P_{I,T_I}] - P_{I,0}, \dots, E_0[P_{I,T_I}] - P_{I,T-1})^{\mathrm{T}} \\ Y &= (P_{0,1} - E_0[P_{0,T_0}], P_{0,2} - E_0[P_{0,T_0}], \dots, P_{0,T} - E_0[P_{0,T_0}], \\ &\qquad \dots, P_{I,1} - E_0[P_{I,T_I}], \dots, P_{I,T} - E_0[P_{I,T_I}])^{\mathrm{T}} \end{aligned}$$

where $(\cdots)^{\mathrm{T}}$ denotes vector transposition. Applying ordinary least squares to the above equation, we can estimate the original OU parameters as follows:

$$\hat{\kappa} = \frac{\mathrm{cov}[Y,X]}{\mathrm{cov}[X,X]}, \qquad \hat{\xi} = Y - \hat{\kappa}X, \qquad \hat{\sigma} = \sqrt{\mathrm{cov}[\hat{\xi},\hat{\xi}]}$$

where, as usual, $\mathrm{cov}[\cdot,\cdot]$ is the covariance operator. We use the above estimations to find optimal stop-loss and take-profit bounds.

Explicit problem formulation

In this rather technical section, we perform transformations in order to formulate the problem in terms of heat potentials. Further details are given in Lipton & Lopez De Prado (2020).

Consider a long investment strategy S and suppose profit/loss opportunity is driven by an OU process (see, for example, Lopez De Prado 2018):

$$\mathrm{d}x' = \kappa'(\theta' - x'), \quad \mathrm{d}t' + \sigma'\,\mathrm{d}W_{t'}, \quad x'(0) = 0 \tag{1}$$

and a trading rule:

$$R = \{\underline{\pi}', \overline{\pi}', T'\}, \quad \underline{\pi}' < 0, \quad \overline{\pi}' > 0$$

Here, x' is the value of the underlying process, θ' is its equilibrium level, κ' is the mean-reversion rate and σ' is the corresponding volatility.

It is important to understand and recognise the natural units associated with the OU process (1). To this end, we can use its steady state. The steady-state expectation of the above process is θ', while its standard deviation is given by:

$$\varOmega' = \sigma'/\sqrt{2\kappa'}$$

As usual, an appropriate scaling is helpful to remove superfluous parameters. To this end, we define:

$$\begin{gathered} t = \kappa' t', \quad T = \kappa' T', \quad x = \frac{\sqrt{\kappa'}}{\sigma'}x' \\ \overline{\pi} = \frac{\sqrt{\kappa'}}{\sigma'}\overline{\pi}', \quad \underline{\pi} = \frac{\sqrt{\kappa'}}{\sigma'}\underline{\pi}' \\ \theta = \frac{\sqrt{\kappa'}}{\sigma'}\theta', \quad E = \frac{E'}{\sqrt{\kappa'}\sigma'}, \quad F = \frac{F'}{\kappa'\sigma'^2} \end{gathered}$$

and get:

$$\mathrm{d}x = (\theta - x)\,\mathrm{d}t + \mathrm{d}W_t$$

in the domain:

$$\underline{\pi} \leqslant x \leqslant \overline{\pi}, \quad 0 \leqslant t \leqslant T$$

Now, all the variables are non-dimensional, and the only non-trivial characteristic of the process is its equilibrium value. The steady-state distribution has an expectation of θ and a standard deviation $\varOmega = 1/\sqrt{2}$.

According to the trading rule, we exit the trade when (a) the price hits $\overline{\pi}$ to take a profit; (b) the price hits $\underline{\pi}$ to stop losses; or (c) the trade expires at $t = T$. For a short investment strategy, the roles of $\{\underline{\pi}, \overline{\pi}\}$ are reversed: profits equal to $-\underline{\pi}$ are taken when the price hits $\underline{\pi}$, and losses equal to $-\overline{\pi}$ are realised when the price hits $\overline{\pi}$.

Intuitively, we go long when $\theta \geqslant 0$ and short when $\theta < 0$. Assuming we know the trading rule $\{\underline{\pi}(\theta,T), \overline{\pi}(\theta,T), T\}$ for $\theta \geqslant 0$, the corresponding trading rule for $\theta < 0$ has the form:

$$\{\underline{\pi}(\theta,T), \overline{\pi}(\theta,T), T\} = \{-\underline{\pi}(-\theta,T), -\overline{\pi}(-\theta,T), T\}$$

Thus, we are interested in the maximisation of the SR for non-negative $\theta \geqslant 0$. We formulate this mathematically below.

For a given T, we define the stopping time:

$$\iota = \inf\{t : x_t = \overline{\pi} \text{ or } x_t = \underline{\pi} \text{ or } t = \underline{T}\}$$

We wish to determine optimal $\overline{\pi} > 0$, $\underline{\pi} < 0$, to maximise the SR:

$$\mathrm{SR} = \frac{\mathbb{E}\{x_\iota/\iota\}}{\sqrt{\mathbb{E}\{x_\iota^2/\iota^2\} - (\mathbb{E}\{x_\iota/\iota\})^2}}$$

We also need to know the expected duration of the trade:

$$\mathrm{DUR} = \mathbb{E}\{\iota\}$$

Let $E(t,x)$, $F(t,x)$ and $G(t,x)$ be the expected return, squared return and duration, respectively. We write the terminal boundary value problem (TBVP) for $E(t,x)$ as follows:

$$\left.\begin{aligned} E_t(t,x) + (\theta - x)E_x(t,x) + \tfrac{1}{2}E_{xx}(t,x) = 0 \\ E(t,\overline{\pi}) = \frac{\overline{\pi}}{t}, \quad E(t,\underline{\pi}) = \frac{\underline{\pi}}{t}, \quad E(T,x) = \frac{x}{T} \end{aligned}\right\} \tag{2}$$

The other TBVPs for $F(t,x)$ and $G(t,x)$ have a similar form. Here, as usual, the subscripts denote partial derivatives.

We can define the SR and the process duration DUR as follows:

$$\mathrm{SR} = \frac{E(0,0)}{\sqrt{F(0,0) - (E(0,0))^2}}$$
$$\mathrm{DUR} = G(0,0)$$

Our objective is to use the method of heat potentials to solve the above TBVPs. It is not possible to do so directly. However, the change of variables:

$$\tau = T - t, \quad \upsilon = \frac{1 - \mathrm{e}^{-2\tau}}{2}, \quad \xi = \mathrm{e}^{-\tau}(x - \theta)$$

comes to the rescue. We prefer to solve the problem backwards, rather than forwards, because it is much more efficient for $T \to \infty$; further details are given in Lipton & Kaushansky (2020a).

We concentrate on (2). Instead of $E(t,x)$, we consider $E(\upsilon,\xi)$, represent this in the form:

$$E(\upsilon,\xi) = \hat{E}(\upsilon,\xi) - \frac{2(\xi+\theta)}{\ln(1-2\Upsilon)}$$

and get the standard initial boundary value problem (IBVP) for the heat equation:

$$\hat{E}_\upsilon(\upsilon,\xi) = \tfrac{1}{2}\hat{E}_{\xi\xi}(\upsilon,\xi)$$
$$\hat{E}(\upsilon,\overline{\Pi}(\upsilon)) = \overline{e}(\upsilon), \quad \hat{E}(\upsilon,\underline{\Pi}(\upsilon)) = \underline{e}(\upsilon), \quad \hat{E}(0,\xi) = 0$$

Here:

$$\underline{e}(\upsilon) = \frac{2\underline{\pi}}{\ln((1-2\upsilon)/(1-2\Upsilon))} + \frac{2(\underline{\Pi}(\upsilon)+\theta)}{\ln(1-2\Upsilon)}$$
$$\overline{e}(\upsilon) = \frac{2\overline{\pi}}{\ln((1-2\upsilon)/(1-2\Upsilon))} + \frac{2(\overline{\Pi}(\upsilon)+\theta)}{\ln(1-2\Upsilon)}$$

By splitting $E(\upsilon,\xi)$ into two parts, we can concentrate on solving a homogeneous IBVP, which is particularly well-suited to being addressed by the method of heat potentials (Tikhonov & Samarskii 1963).

We can also derive similar IBVPs for $\hat{F}(\upsilon,\xi)$, $\hat{G}(\upsilon,\xi)$, where:

$$\hat{F}(\upsilon,\xi) = F(\upsilon,\xi) - \frac{4(\upsilon + (\xi+\theta)^2)}{(\ln(1-2\Upsilon))^2}$$
$$\hat{G}(\upsilon,\xi) = G(\upsilon,\xi) + \tfrac{1}{2}\ln(1-2\Upsilon)$$

These IBVPs have the form:

$$\hat{F}_\upsilon(\upsilon,\xi) = \tfrac{1}{2}\hat{F}_{\xi\xi}(\upsilon,\xi)$$
$$\hat{F}(\upsilon,\overline{\Pi}(\upsilon)) = \overline{f}(\upsilon), \quad \hat{F}(\upsilon,\underline{\Pi}(\upsilon)) = \underline{f}(\upsilon), \quad \hat{F}(0,\xi) = 0$$

and:

$$\hat{G}_\upsilon(\upsilon,\xi) = \tfrac{1}{2}\hat{G}_{\xi\xi}(\upsilon,\xi)$$
$$\hat{G}(\upsilon,\overline{\Pi}(\upsilon)) = \overline{g}(\upsilon), \quad \hat{G}(\upsilon,\underline{\Pi}(\upsilon)) = \underline{g}(\upsilon), \quad \hat{G}(0,\xi) = 0$$

where:

$$\overline{f}(\upsilon) = \frac{4\overline{\pi}^2}{(\ln((1-2\upsilon)/(1-2\Upsilon)))^2} - \frac{4(\upsilon + (\overline{\Pi}(\upsilon)+\theta)^2)}{(\ln(1-2\Upsilon))^2}$$
$$\underline{f}(\upsilon) = \frac{4\underline{\pi}^2}{(\ln((1-2\upsilon)/(1-2\Upsilon)))^2} - \frac{4(\upsilon + (\underline{\Pi}(\upsilon)+\theta)^2)}{(\ln(1-2\Upsilon))^2}$$
$$\overline{g}(\upsilon) = \tfrac{1}{2}\ln(1-2\upsilon), \quad \underline{g}(\upsilon) = \tfrac{1}{2}\ln(1-2\upsilon)$$

As a result, we can represent the quantities of interest in terms of $\hat{E}(\Upsilon,\varpi)$, $\hat{F}(\Upsilon,\varpi)$ and $\hat{G}(\Upsilon,\varpi)$:

$$\mathrm{SR} = \frac{\hat{E}(\Upsilon,\varpi) - (2(\varpi+\theta)/\ln(1-2\Upsilon))}{\sqrt{\hat{F}(\Upsilon,\varpi) - (\hat{E}(\Upsilon,\varpi))^2 + \frac{4(\Upsilon + \ln(1-2\Upsilon)(\varpi+\theta)\hat{E}(\Upsilon,\varpi))}{(\ln(1-2\Upsilon))^2}}} \tag{3}$$

$$\mathrm{DUR} = \hat{G}(\Upsilon,\varpi) - \tfrac{1}{2}\ln(1-2\Upsilon) \tag{4}$$

Here:

$$\Upsilon = \frac{1-\mathrm{e}^{-2T}}{2}, \quad \varpi = -\sqrt{1-2\Upsilon}\,\theta$$
$$\overline{\Pi}(\upsilon) = \sqrt{1-2\upsilon}(\overline{\pi}-\theta), \quad \underline{\Pi}(\upsilon) = \sqrt{1-2\upsilon}(\underline{\pi}-\theta)$$

Thus, after the above transformations, the problem becomes solvable by the method of heat potentials.

The method of heat potentials

Now we are ready to use the classical method of heat potentials to calculate the SR. Assume $\hat{E}$, $\hat{F}$ and $\hat{G}$ can be treated by the same token. To find $\hat{E}$, one needs to solve the following coupled system of Volterra integral equations (see, for example, Lipton 2001; Tikhonov & Samarskii 1963):

$$\begin{aligned}\underline{\varepsilon}(\upsilon) &+ \frac{1}{\sqrt{2\pi}}\int_0^\upsilon \frac{(\underline{\Pi}(\upsilon)-\underline{\Pi}(\zeta))\exp(-\frac{(\underline{\Pi}(\upsilon)-\underline{\Pi}(\zeta))^2}{2(\upsilon-\zeta)})}{(\upsilon-\zeta)^{3/2}}\underline{\varepsilon}(\zeta)\,\mathrm{d}\zeta \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^\upsilon \frac{(\underline{\Pi}(\upsilon)-\overline{\Pi}(\zeta))\exp(-\frac{(\underline{\Pi}(\upsilon)-\overline{\Pi}(\zeta))^2}{2(\upsilon-\zeta)})}{(\upsilon-\zeta)^{3/2}}\overline{\varepsilon}(\zeta)\,\mathrm{d}\zeta = \underline{e}(\upsilon)\end{aligned} \tag{5}$$

$$\begin{aligned}-\overline{\varepsilon}(\upsilon) &+ \frac{1}{\sqrt{2\pi}}\int_0^\upsilon \frac{(\overline{\Pi}(\upsilon)-\underline{\Pi}(\zeta))\exp(-\frac{(\overline{\Pi}(\upsilon)-\underline{\Pi}(\zeta))^2}{2(\upsilon-\zeta)})}{(\upsilon-\zeta)^{3/2}}\underline{\varepsilon}(\zeta)\,\mathrm{d}\zeta \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^\upsilon \frac{(\overline{\Pi}(\upsilon)-\overline{\Pi}(\zeta))\exp(-\frac{(\overline{\Pi}(\upsilon)-\overline{\Pi}(\zeta))^2}{2(\upsilon-\zeta)})}{(\upsilon-\zeta)^{3/2}}\overline{\varepsilon}(\zeta)\,\mathrm{d}\zeta = \overline{e}(\upsilon)\end{aligned} \tag{6}$$

The corresponding functions for $\hat{F}$ and $\hat{G}$ are denoted by $(\underline{\phi}(\zeta),\overline{\phi}(\zeta))$ and $(\underline{\gamma}(\zeta),\overline{\gamma}(\zeta))$, respectively.

Once (5) and (6) are solved, $\hat{E}(\upsilon,\xi)$ can be written as follows:

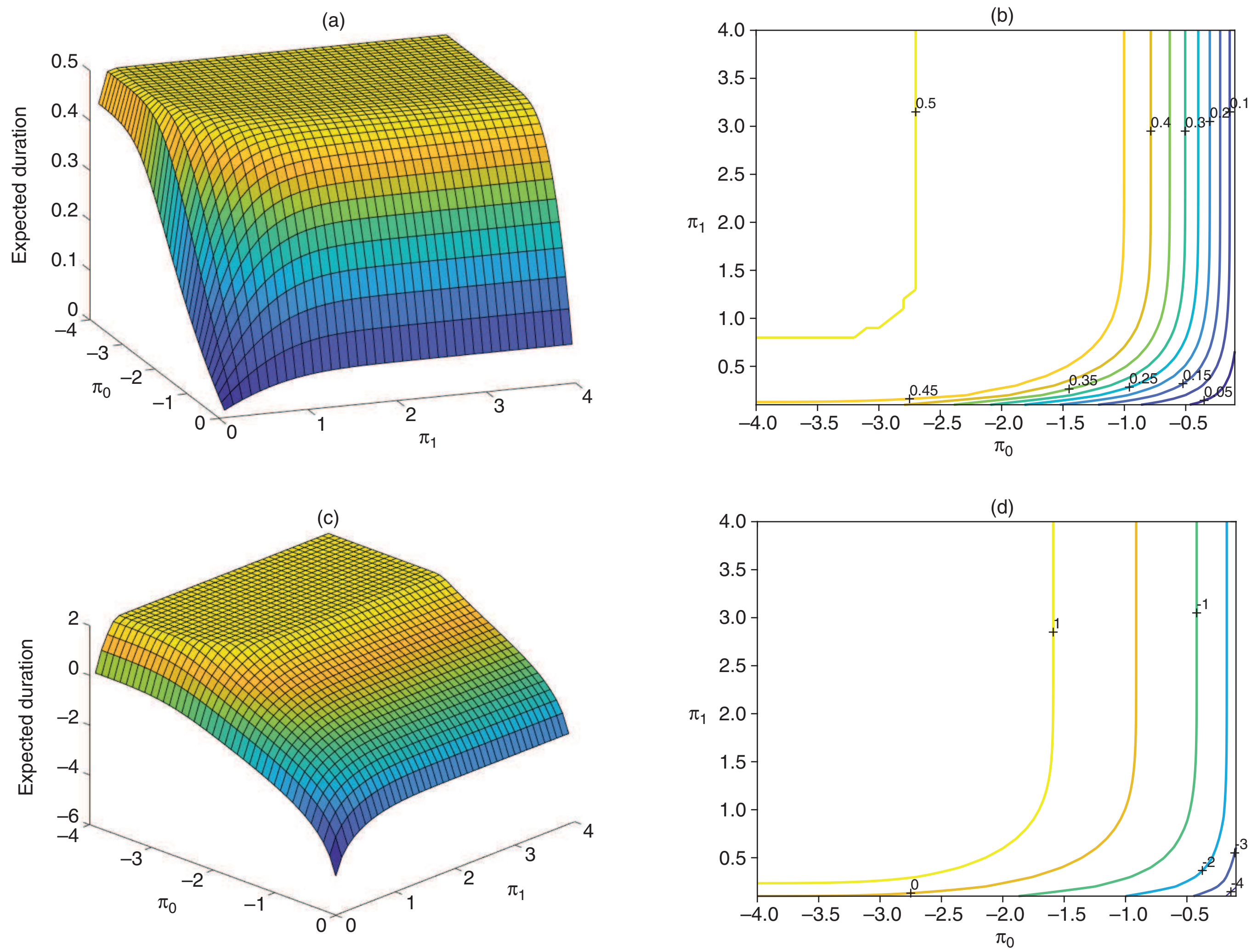
$$\begin{aligned}\hat{E}(\upsilon,\xi) = &\frac{1}{\sqrt{2\pi}}\int_0^\upsilon \frac{(\xi-\underline{\Pi}(\zeta))\exp(-\frac{(\xi-\underline{\Pi}(\zeta))^2}{2(\upsilon-\zeta)})}{(\upsilon-\zeta)^{3/2}}\underline{\varepsilon}(\zeta)\,\mathrm{d}\zeta \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^\upsilon \frac{(\xi-\overline{\Pi}(\zeta))\exp(-\frac{(\xi-\overline{\Pi}(\zeta))^2}{2(\upsilon-\zeta)})}{(\upsilon-\zeta)^{3/2}}\overline{\varepsilon}(\zeta)\,\mathrm{d}\zeta\end{aligned} \tag{7}$$

In particular:

$$\begin{aligned}\hat{E}(\Upsilon,\varpi) = &\frac{1}{\sqrt{2\pi}}\int_0^\Upsilon \frac{(\varpi-\underline{\Pi}(\zeta))\exp(-\frac{(\varpi-\underline{\Pi}(\zeta))^2}{2(\Upsilon-\zeta)})}{(\Upsilon-\zeta)^{3/2}}\underline{\varepsilon}(\zeta)\,\mathrm{d}\zeta \\ &+ \frac{1}{\sqrt{2\pi}}\int_0^\Upsilon \frac{(\varpi-\overline{\Pi}(\zeta))\exp(-\frac{(\varpi-\overline{\Pi}(\zeta))^2}{2(\Upsilon-\zeta)})}{(\Upsilon-\zeta)^{3/2}}\overline{\varepsilon}(\zeta)\,\mathrm{d}\zeta\end{aligned}$$

It is important to note that $(\underline{\varepsilon}(\zeta),\overline{\varepsilon}(\zeta))$ are singular at $\zeta = \Upsilon$. However, due to the dampening impact of the exponents $\exp(-(\varpi - \overline{\Pi}(\zeta))^2/2(\Upsilon-\zeta))$, the corresponding integrals still converge.

1 (a), (b) The expected duration $\Upsilon = (1 - \exp(-2G)/2)$ as a function of $(\underline{\pi}, \overline{\pi})$; (c), (d) the logarithm of the expected duration G



The corresponding $\theta = 1.0$. Here, and in figures 2 and 3, $\pi_0 \equiv \underline{\pi}$ and $\pi_1 \equiv \overline{\pi}$

Once we know $\hat{E}(\Upsilon, \varpi)$, $\hat{F}(\Upsilon, \varpi)$ and $\hat{G}(\Upsilon, \varpi)$, the SR and DUR can be calculated using (3) and (4).

Numerical method

To compute the SR, we need to find $\hat{E}(\Upsilon, \varpi)$ and $\hat{F}(\Upsilon, \varpi)$, and then apply (3). $\hat{E}(\Upsilon, \varpi)$ can be found from equation (7) by simple integration with pre-computed $(\underline{\varepsilon}, \overline{\varepsilon})$. In this section, we develop a numerical method to compute these quantities by solving (5) and (6), by extending the method described in Lipton & Kaushansky (2020a). For illustrative purposes, we develop a simple scheme based on the trapezoidal rule for Stieltjes integrals.

We want to solve a generic system of the form:

$$\nu^1(\upsilon) + \int_0^\upsilon \frac{K^{1,1}(\upsilon, s)}{\sqrt{\upsilon - s}} \nu^1(s)\,\mathrm{d}s + \int_0^\upsilon K^{1,2}(\upsilon, s)\nu^2(s)\,\mathrm{d}s = \chi^1(\upsilon)$$

$$-\nu^2(\upsilon) + \int_0^\upsilon K^{2,1}(\upsilon, s)\nu^1(s)\,\mathrm{d}s + \int_0^\upsilon \frac{K^{2,2}(\upsilon, s)}{\sqrt{\upsilon - s}} \nu^2(s)\,\mathrm{d}s = \chi^2(\upsilon)$$

with respect to variables $(\nu^1(\upsilon), \nu^2(\upsilon))$, where:

$$K^{i,j}(\upsilon, s) = \frac{1}{\sqrt{2\pi}} \frac{\Xi^{i,i}(\upsilon, s)}{(\upsilon - s)} \exp\left(-\frac{(\Xi^{i,i}(\upsilon, s))^2}{2(\upsilon - s)}\right), \quad i = j$$

$$K^{i,j}(\upsilon, s) = \frac{1}{\sqrt{2\pi}} \frac{\Xi^{i,j}(\upsilon, s)}{(\upsilon - s)^{3/2}} \exp\left(-\frac{(\Xi^{i,j}(\upsilon, s))^2}{2(\upsilon - s)}\right), \quad i \neq j$$

Here:

$$\Xi^{1,1}(\upsilon, s) = \underline{\Pi}(\upsilon) - \underline{\Pi}(s), \quad \Xi^{1,2}(\upsilon, s) = \underline{\Pi}(\upsilon) - \overline{\Pi}(s)$$
$$\Xi^{2,1}(\upsilon, s) = \overline{\Pi}(\upsilon) - \underline{\Pi}(s), \quad \Xi^{2,2}(\upsilon, s) = \overline{\Pi}(\upsilon) - \overline{\Pi}(s)$$

It is clear that:

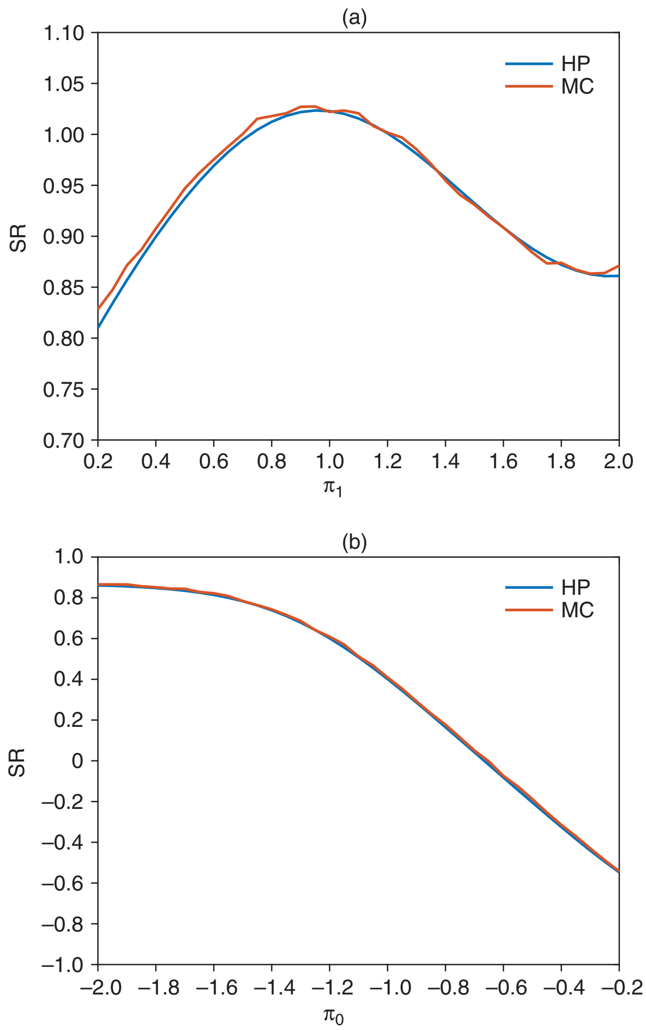
$$K^{1,1}(\upsilon, \upsilon) = \frac{\theta - \underline{\pi}}{\sqrt{2\pi}\sqrt{1 - 2\upsilon}}, \quad K^{1,2}(\upsilon, \upsilon) = 0$$

$$K^{2,1}(\upsilon, \upsilon) = 0, \quad K^{2,2}(\upsilon, \upsilon) = \frac{\theta - \overline{\pi}}{\sqrt{2\pi}\sqrt{1 - 2\upsilon}}$$

Consider a grid $0 = \upsilon_0 < \upsilon_1 < \cdots < \upsilon_n = \Upsilon$, and let:

$$\Delta_{k,l} = \upsilon_k - \upsilon_l$$

2 (a) The SR as a function of π_1 computed by using the method of heat potentials and the Monte Carlo method for $\pi_0 = -1$; (b) the SR as a function of π_0 computed using the method of heat potentials and the Monte Carlo method for $\pi_1 = 1$



$\theta = 1.0$, $T = 1.96$

We can rewrite the relevant integrals as Stieltjes integrals and, using the trapezoidal rule for approximation of integrals, get the following expression for (ν_k^1, ν_k^2):

$$\nu_k^1 + \sum_{i=1}^{k}\Bigg(\frac{(K_{k,i}^{1,1}\nu_i^1 + K_{k,i-1}^{1,1}\nu_{i-1}^1)}{(\sqrt{\Delta_{k,i}} + \sqrt{\Delta_{k,i-1}})} + \tfrac{1}{2}(K_{k,i}^{1,2}\nu_i^2 + K_{k,i-1}^{1,2}\nu_{i-1}^2)\Bigg)\Delta_{i,i-1} = \chi_k^1$$

$$-\nu_k^2 + \sum_{i=1}^{k}\Bigg(\tfrac{1}{2}(K_{k,i}^{2,1}\nu_i^1 + K_{k,i-1}^{2,1}\nu_{i-1}^1) + \frac{(K_{k,i}^{2,2}\nu_i^2 + K_{k,i-1}^{2,2}\nu_{i-1}^2)}{(\sqrt{\Delta_{k,i}} + \sqrt{\Delta_{k,i-1}})}\Bigg)\Delta_{i,i-1} = \chi_k^2$$

where:

$$\nu_i^\alpha = \nu^\alpha(\upsilon_i), \quad \chi_i^\alpha = \chi^\alpha(\upsilon_i), \quad K_{k,j}^{\alpha,\beta} = K^{\alpha,\beta}(\upsilon_k, \upsilon_i), \quad \alpha, \beta = 1, 2$$

The approximation error of the integrals is of order $O(\Delta^2)$, where $\Delta = \max_i(\Delta_{i,i-1})$. Hence, on a uniform grid, the convergence is of order $O(\Delta)$. We emphasise that, due to the nature of $(\underline{e}(\upsilon), \overline{e}(\upsilon))$, etc, it is necessary to use a highly inhomogeneous grid that is concentrated near the right end-point.

■ **Computation of the SR.** Once $(\underline{\varepsilon}(\upsilon), \overline{\varepsilon}(\upsilon))$ are computed, we can approximate $\hat{E}(\upsilon, \xi)$. We are interested in computing these functions at one point (Υ, ϖ), which can be done by approximation of the integrals using the trapezoidal rule:

$$\hat{E}(\Upsilon, \varpi) = \frac{1}{2}\sum_{i=1}^{k}(\underline{w}_{n,i}\underline{\varepsilon}_i + \underline{w}_{n,i-1}\underline{\varepsilon}_{i-1} + \overline{w}_{n,i}\overline{\varepsilon}_i + \overline{w}_{n,i-1}\overline{\varepsilon}_{i-1})\Delta_{i,i-1} \tag{8}$$

where the corresponding weights are as follows:

$$\underline{w}_{n,i} = \frac{(\varpi - \underline{\Pi}_i)\exp(-\frac{(\varpi - \underline{\Pi}_i)^2}{2\Delta_{n,i}})}{\sqrt{2\pi}\Delta_{n,i}^{3/2}}$$

$$\overline{w}_{n,i} = \frac{(\varpi - \overline{\Pi}_i)\exp(-\frac{(\varpi - \overline{\Pi}_i)^2}{2\Delta_{n,i}})}{\sqrt{2\pi}\Delta_{n,i}^{3/2}}, \quad 1 \leqslant i < n$$

$$\underline{w}_{n,i} = 0, \quad \overline{w}_{n,i} = 0, \quad i = n$$

As a result, we get the following algorithm for the numerical evaluation of the SR.

Algorithm 1 Numerical evaluation of the SR

Step 1. Define a time grid $0 = \upsilon_0 < \upsilon_1 < \ldots < \Upsilon$.

Step 2. Compute $\underline{\epsilon}(\upsilon)$, $\overline{\epsilon}(\upsilon)$, $\underline{\phi}(\upsilon)$ and $\overline{\phi}(\upsilon)$ using the numerical method in the 'Numerical method' section above.

Step 3. Compute $\hat{E}(\Upsilon, \varpi)$ by using (8).

Step 4. Compute $\hat{F}(\Upsilon, \varpi)$ by the same token.

Step 5. Compute the SR by using (3).

Expected duration of the trade

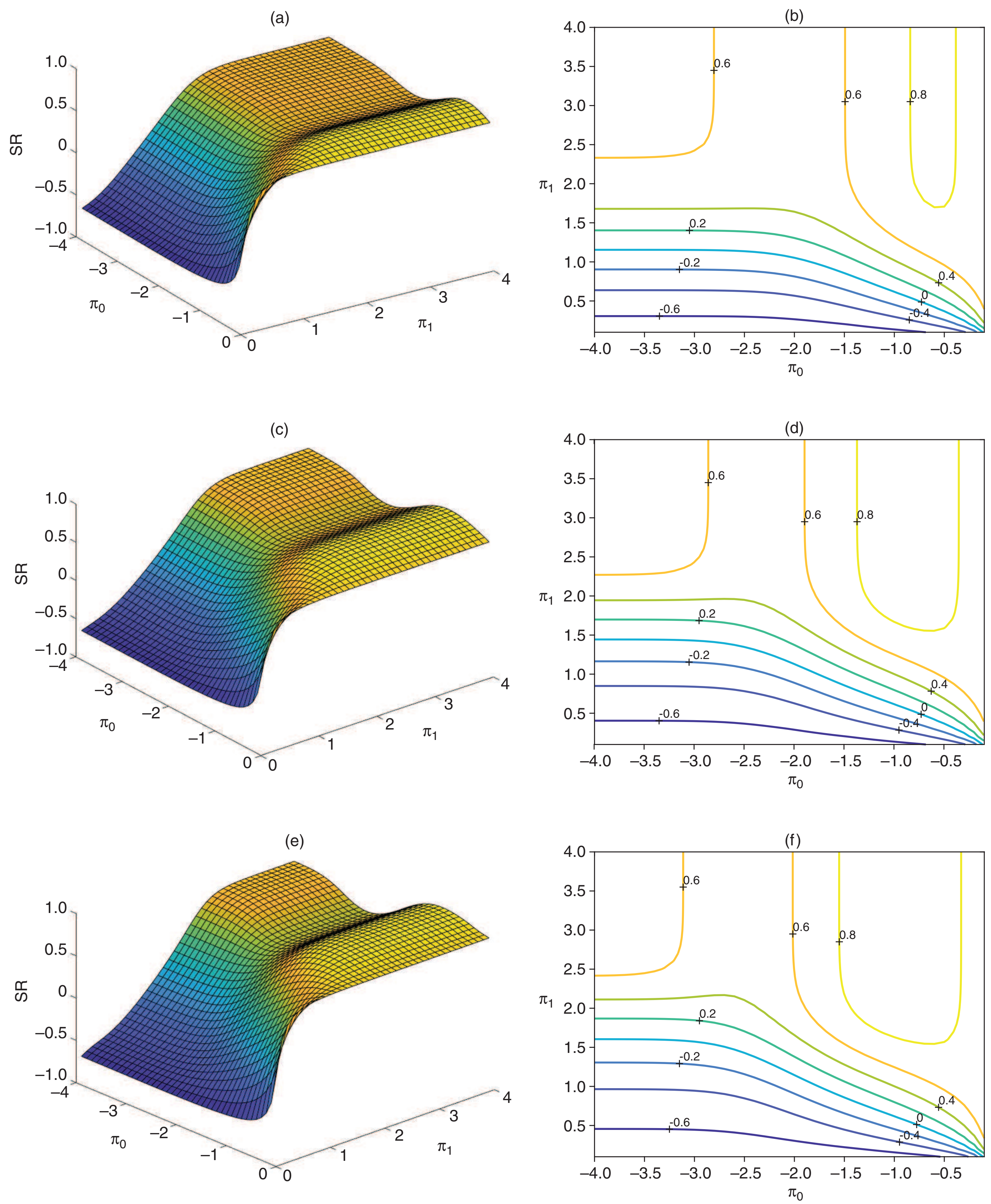
The method of heat potentials boils down to solving a system of Volterra equations of the second kind. However, there are particular quantities of interest that can be calculated directly. One such quantity is the expected value of the duration of a trade, which terminates when the spread hits one of the barriers, $T = \infty$ (or $\Upsilon = 0.5$). This quantity can be found analytically by solving an inhomogeneous linear ordinary differential equation. We show the expected duration as a function of $\underline{\pi}, \overline{\pi}$ for $\theta = 1$ in figure 1.

Given that $\Upsilon \to 0.5$ corresponds to $T \to \infty$, we can see from this figure that for sufficiently remote $\underline{\pi}, \overline{\pi}$ the process stays within the range $[\underline{\pi}, \overline{\pi}]$ indefinitely – or, at least, for a very long time.

Numerical results

■ **Comparison with Monte Carlo simulations.** We compute the SR for various values of $\underline{\pi}$ and $\overline{\pi}$, and as a result show the SR as a function of $(\underline{\pi}, \overline{\pi})$. Then one can choose $(\underline{\pi}, \overline{\pi})$ in order to maximise the SR.

3 The SR as a function of (π_0, π_1) for $\theta = 0.5$ and (a), (b) $T = 1.96$, (c), (d) $T = 4.26$, and (e), (f) $T = 6.56$



A. The SR maximised over $(\underline{\pi}, \overline{\pi})$ for fixed Υ or T

	Υ, T		
θ	0.49, 0.8	0.4999, 1.96	0.499999, 4.26
1.0	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 4.0$ SR = 1.2261	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 4.0$ SR = 1.3824	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 4.0$ SR = 1.3709
0.5	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.6$ SR = 0.8219	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.9$ SR = 0.8792	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 1.0$ SR = 0.8963
0.0	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.1$ SR = 0.7075	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.4$ SR = 0.7139	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.1$ SR = 0.7411

Consider $\theta = 1.0$ and $\Upsilon = 0.49$, $T = 1.96$. To ensure the validity of our derivations, we compare our numerical results with the Monte Carlo method, which simulates the process and computes its expectation and variance (see Lopez De Prado 2018). We show the results for the SR in figure 2.

We see that the relative difference between the method of heat potentials and the Monte Carlo method is small and mainly comes from the Monte Carlo noise.

■ **Optimisation of the SR.** In this section, we solve a problem of finding parameters to maximise the SR by analysing it as a function of $(\underline{\pi}, \overline{\pi})$ for different values of θ and Υ. Two problems are considered: (a) fix Υ and maximise the SR over $(\underline{\pi}, \overline{\pi})$; and (b) maximise the SR over $(\underline{\pi}, \overline{\pi}, \Upsilon)$.

Given the natural unit $\Omega = 1/\sqrt{2}$, we consider three representative values of θ – namely, $\theta = 1$, $\theta = 0.5$ and $\theta = 0$ – corresponding to strong and weak mispricing and fair pricing, respectively. We choose three maturities, $\Upsilon = 0.49$, 0.4999 and 0.499999, or, equivalently, $T = 1.96$, 4.26 and 6.56. For negative θ, the corresponding SR can be obtained by reflection if needed. The optimal bounds $(\underline{\pi}^*, \overline{\pi}^*)$ are given in table A.

Table A shows that when the original mispricing is strong ($\theta = 1$) it is not optimal to stop the trade early. When the mispricing is weaker ($\theta = 0.5$) or there is no mispricing in the first place ($\theta = 0$) it is not optimal to stop losses, but it might be beneficial to take profits. In practice, one needs to use a highly reliable estimation of the OU parameters to employ these rules with confidence. For brevity, we show the corresponding SR surface only for $\theta = 0.5$ in figure 3.

Conclusions

In this article, we create an analytical framework for computing optimal stop-loss/take-profit bounds $(\underline{\pi}^*, \overline{\pi}^*)$ for OU driven trading strategies by using the method of heat potentials.

First, we present a method for calibrating the corresponding OU process to market prices. Second, we derive an explicit expression for the SR given by (3) and maximise it with respect to the stop-loss/take-profit bounds $(\underline{\pi}, \overline{\pi})$. Third, for three representative values of θ, we calculate the SR on a grid of $(\underline{\pi}, \overline{\pi})$, perform an optimisation and present $(\underline{\pi}^*, \overline{\pi}^*)$ in table A. We graphically summarise results for $\theta = 0.5$ in figure 3. For strong mispricing, in agreement with intuition, it is optimal to wait until the trade's expiration without imposing stop-loss/take-profit bounds. For weaker mispricing, it is not optimal to stop losses, but it might be optimal to take profits early. Still, to be on the safe side, we recommend imposing stop losses chosen following one's risk appetite to avoid unpleasant surprises caused by misspecification of the underlying process.

Our rules help liquidity providers to decide how to offer liquidity to the market in the most profitable way and help statistical arbitrage traders to execute their trading strategies optimally.

We shall discuss a new and challenging multi-dimensional version of these rules (covering several correlated stocks) elsewhere. ■

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