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OLD PROBLEMS, CLASSICAL METHODS, NEW SOLUTIONS

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We use a powerful extension of the classical method of heat potentials, recently developed by the present author and his collaborators, to solve several significant problems of financial mathematics. We consider the following problems in detail: (a) calibrating the default boundary in the structural default framework to a constant default intensity; (b) calculating default probability for a representative bank in the mean-field framework; and (c) finding the hitting time probability density of an Ornstein–Uhlenbeck process. Several other problems, including pricing American put options and finding optimal mean-reverting trading strategies, are mentioned in passing. Besides, two nonfinancial applications — the supercooled Stefan problem and the integrate-and-fire neuroscience problem — are briefly discussed as well.

Keywords: Method of heat potentials; first hitting time density; Cherkasov condition; Volterra integral equation; Abel integral equation; Ornstein–Uhlenbeck process; pairs trading; structural default model; stability of banking system; Stefan problem; integrate-and-fire neuron excitation model.

1. Introduction

The method of heat potentials (MHP) is a highly robust and versatile approach frequently exploited in mathematical physics; see e.g. Tikhonov & Samarskii (1963), Rubinstein (1971), Kartashov (2001) and Watson (2012), among others. It is essential in numerous vital fields, such as thermal engineering, nuclear engineering, and material science. However, it is not particularly well known in mathematical finance, even though the first meaningful use in this context was described by the present author almost 20 years ago. The specific application was to pricing barrier options with curvilinear barriers; see Sec. 12.2.3 in Lipton (2001) .

In this paper, we demonstrate how a powerful extension of the classical MHP, recently developed by the present author and his collaborators, can be used to solve seemingly unrelated problems of applied mathematics in general and financial mathematics in particular; see Lipton & Kaushansky (2018, 2020a, 2020b), Lipton & Lopez de Prado (2020), and Lipton *et al.* (2019). Specifically, we use the extended method of heat potentials (EMHP) for (a) calibrating the default boundary for a structural default model with constant default intensity, (b) finding a semi-analytical solution of the mean-field problem for a system of interacting banks, and (c) developing a semi-analytical description for the hitting time density for an Ornstein–Uhlenbeck (OU) process. Besides, we demonstrate the efficacy of EMHP by considering two nonfinancial applications: (a) the supercooled Stefan problem and (b) the integrate-and-fire model in neuroscience.

We note in passing that, in addition to the problems discussed in this paper, the EMHP has been successfully used for pricing American put options and for finding optimal strategies for mean-reverting spread trading; see Lipton & Kaushansky (2020a, 2020b).

We emphasize that in most cases, the EMHP beats all other known approaches to the problem in question, and in some instances, for example, for the boundary calibration problem, it is the only one that can be used effectively.

The paper is organized as follows. In Sec. 2, we present the mathematical preliminaries regarding the classical MHP and describe its useful extensions and generalizations, which we dub the EMHP. In Sec. 3, we examine the structural default problem. In Sec. 4, we study the mean-field banking system in the structural default framework and analyze its stability and resilience. In Sec. 5, we describe the EMHP approach to calculating the hitting time probability distribution for an Ornstein–Uhlenbeck process. The EMHP turns to be a powerful and versatile tool, which solves this complicated problem in its entirety. In Secs. 6 and 7, we consider two nonfinancial applications of the EMHP — the supercooled Stefan problem and the integrate-and-fire neuron excitation model. We draw our conclusions in Sec. 8.

2. Mathematical Preliminaries

In this section, we describe the classical MHP and its beneficial extensions proposed by the present author and his collaborators. The MHP is uniquely well suited to solving rather challenging problems occurring routinely in applied mathematics in general, and in financial engineering in particular. In a nutshell, this method allows one to reduce a complicated partial differential equation of the parabolic type with a time-dependent boundary to a much simpler Volterra integral equation.

2.1. *The method of heat potentials*

Consider a standard heat equation in a one-sided domain with a moving boundary $b^>(t)$:

$$\begin{aligned} &\frac{\partial}{\partial t}E^>(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}E^>(t,x), \quad b^>(t) \le x < \infty, \\ &E^>(0,x) = \varepsilon^>(x), \quad E^>(t,b^>(t)) = e^>(t), \quad E(t,x \to \infty) \to 0. \end{aligned} \tag{1}$$

Here and below, we use the superscript $>$ ($<$) to emphasize the fact that we are interested in the computational domain limited by the boundary from below (above). Without loss of generality, we can assume that $\varepsilon^>(x) = 0$; the case of a nonzero initial condition can be solved by splitting

$$E^> = E(t,x) + F^>(t,x), \tag{2}$$

$$E(t,x) = \int_{b(t)}^{\infty} H(t,x-y)\varepsilon^>(y)dy, \tag{3}$$

where $H(t,x)$ is the standard heat kernel,

$$H(t,x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}. \tag{4}$$

Thus, we can restrict ourselves to the case of zero initial condition:

$$\begin{aligned} &\frac{\partial}{\partial t}F^>(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}F^>(t,x), \quad b^>(t) \le x < \infty, \\ &F^>(0,x) = 0, \quad F^>(t,b^>(t)) = f^>(t), \quad F(t,x \to \infty) \to 0, \end{aligned} \tag{5}$$

where

$$f^>(t) = e^>(t) - E(t,b(t)). \tag{6}$$

The MHP allows one to represent $F^>(t,x)$ in the form

$$F^>(t,x) = \int_0^t \frac{(x-b^>(t'))\exp\left(-\frac{(x-b^>(t'))^2}{2(t-t')}\right)\nu^>(t')}{\sqrt{2\pi(t-t')^3}}dt', \tag{7}$$

where $\nu^>(t)$ solves the Volterra equation of the second kind:

$$\nu^>(t') + \int_0^t \frac{\Theta^>(t,t')\Xi^>(t,t')\nu^>(t')}{\sqrt{2\pi(t-t')}}dt' = f^>(t), \tag{8}$$

and

$$\Theta^{>}(t,t') = \frac{b^{>}(t) - b^{>}(t')}{(t-t')}, \quad \Xi^{>}(t,t') = e^{-\frac{(t-t')\Theta^{>2}(t,t')}{2}},$$
$$\Theta^{>}(t,t) = \frac{db^{>}(t)}{dt}, \quad \Xi^{>}(t,t) = 1. \tag{9}$$

Similarly, the solution to the problem

$$\frac{\partial}{\partial t}F^{<}(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}F^{<}(t,x), \quad \infty < \ x \le b^{<}(t),$$
$$F^{<}(0,x) = 0, \quad F^{<}(t, x \to -\infty) \to 0, \quad F^{<}(t, b(t)) = f^{<}(t), \tag{10}$$

has the form

$$F^{<}(t,x) = \int_0^t \frac{(x - b^{<}(t'))\exp\left(-\frac{(x-b^{<}(t'))^2}{2(t-t')}\right)\nu^{<}(t')}{\sqrt{2\pi(t-t')^3}}dt', \tag{11}$$

where

$$-\nu^{<}(t') + \int_0^t \frac{\Theta^{<}(t,t')\Xi^{<}(t,t')\nu^{<}(t')}{\sqrt{2\pi(t-t')}}dt' = f^{<}(t). \tag{12}$$

Finally, the solution to the two-sided problem

$$\frac{\partial}{\partial t}F^{><}(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}F^{><}(t,x), \quad b^{>}(t) \le \ x \le b^{<}(t),$$
$$F^{><}(0,x) = 0, \quad F^{><}(t, b^{<}(t)) = f^{<}(t), \quad F^{><}(t, b^{>}(t)) = f^{>}(t), \tag{13}$$

has the form

$$F^{><}(t,x) = \int_0^t \frac{(x - b^{>}(t'))\exp\left(-\frac{(x-b^{>}(t'))^2}{2(t-t')}\right)\nu^{>}(t')}{\sqrt{2\pi(t-t')^3}}dt'$$
$$+ \int_0^t \frac{(x - b^{<}(t'))\exp\left(-\frac{(x-b^{<}(t'))^2}{2(t-t')}\right)\nu^{<}(t')}{\sqrt{2\pi(t-t')^3}}dt', \tag{14}$$

$$\nu^{>}(t') + \int_0^t \frac{\Theta^{\gg}(t,t')\Xi^{\gg}(t,t')\nu^{>}(t')}{\sqrt{2\pi(t-t')}}dt' + \int_0^t \frac{\Theta^{><}(t,t')\Xi^{><}(t,t')\nu^{<}(t')}{\sqrt{2\pi(t-t')}}dt'$$
$$= f^{>}(t),$$
$$-\nu^{<}(t') + \int_0^t \frac{\Theta^{<>}(t,t')\Xi^{<>}(t,t')\nu^{>}(t')}{\sqrt{2\pi(t-t')}}dt' + \int_0^t \frac{\Theta^{\ll}(t,t')\Xi^{\ll}(t,t')\nu^{<}(t')}{\sqrt{2\pi(t-t')}}dt'$$
$$= f^{<}(t), \tag{15}$$

where

$$\Theta^{\gg}(t,t') = \frac{b^{>}(t) - b^{>}(t')}{(t-t')}, \quad \Theta^{><}(t,t') = \frac{b^{>}(t) - b^{<}(t')}{(t-t')}, \tag{16}$$

and so on.

2.2. *Extensions*

While Eqs. (7), (8) provide an elegant solution to problem (1), in many instances we are interested in the behavior of this solution on the boundary itself. For instance, in numerous problems of mathematical finance, some of which are described below, what we need to know is the function

$$g^{>}(t) = \frac{1}{2}\frac{\partial}{\partial x}F^{>}(t, b^{>}(t)), \tag{17}$$

which represent the outflow of probability from the computational domain. This function can be calculated in two ways.

On the one hand, we can integrate the heat equation and get

$$\begin{aligned}
\frac{d}{dt}\int_{b^{>}(t)}^{\infty} F^{>}(t,x)dx &= \int_{b^{>}(t)}^{\infty} \frac{\partial}{\partial t}F^{>}(t,x)dx - \frac{db^{>}(t)}{dt}f^{>}(t) \\
&= \frac{1}{2}\int_{b^{>}(t)}^{\infty} \frac{\partial^2}{\partial x^2}F^{>}(t,x)dx - \frac{db^{>}(t)}{dt}f^{>}(t) \\
&= -\frac{1}{2}\frac{\partial}{\partial x}F^{>}(t,b^{>}(t)) - \frac{db^{>}(t)}{dt}f^{>}(t) \\
&= -g^{>}(t) - \frac{db^{>}(t)}{dt}\left(\nu^{>}(t) + \int_0^t \frac{\Theta^{>}(t,t')\Xi^{>}(t,t')\nu^{>}(t')}{\sqrt{2\pi(t-t')}}dt'\right).
\end{aligned} \tag{18}$$

Eq. (7) yields

$$\int_{b^{>}(t)}^{\infty} F^{>}(t,x)dx = \int_0^t \frac{\Xi^{>}(t,t')\nu^{>}(t')}{\sqrt{2\pi(t-t')}}dt', \tag{19}$$

so that

$$\begin{aligned}
g^{>}(t) = &-\frac{d}{dt}\int_0^t \frac{\Xi^{>}(t,t')\nu^{>}(t')}{\sqrt{2\pi(t-t')}}dt' - \frac{db^{>}(t)}{dt}\Bigg(\nu^{>}(t) \\
&+ \int_0^t \frac{\Theta^{>}(t,t')\Xi^{>}(t,t')\nu^{>}(t')}{\sqrt{2\pi(t-t')}}dt'\Bigg).
\end{aligned} \tag{20}$$

On the other hand, a useful formula derived by the present author and his collaborators; see Lipton & Kaushansky (2018, 2020a,b), Lipton *et al.* (2019), gives an alternative expression for $g^{>}(t)$:

$$g^{>}(t) = -\left(\frac{1}{\sqrt{2\pi t}} + \frac{db^{>}(t)}{dt}\right)\nu^{>}(t) - \frac{1}{2}\int_0^t \frac{(\Phi^{>}(t,t') + \Theta^{>2}(t,t')\Xi^{>}(t,t')\nu^{>}(t'))}{\sqrt{2\pi(t-t')}}dt', \tag{21}$$

where

$$\Phi^{>}(t,t') = \frac{(\nu^{>}(t) - \Xi^{>}(t,t')\nu^{>}(t'))}{(t-t')}, \quad \Phi^{>}(t,t) = \frac{d\nu^{>}(t)}{dt} + \frac{1}{2}\left(\frac{db^{>}(t)}{dt}\right)^2 \nu^{>}(t). \tag{22}$$

On the surface, Eqs. (20), (21) look very different. However, a useful Lemma proven in Lipton *et al.* (2019), allows one to connect the two.

Lemma. *Let* $\Psi(t,t')$ *be a differentiable function, such that* $\Psi(t,t) = 1$. *Then*

$$\frac{d}{dt}\int_0^t \frac{\Psi(t,t')\,\nu(t')}{\sqrt{2\pi(t-t')}}dt' = \frac{\nu(t)}{\sqrt{2\pi t}} + \frac{1}{2}\int_0^t \frac{\nu(t) - (\Psi(t,t') - 2(t-t')\Psi_t(t,t'))\,\nu(t')}{\sqrt{2\pi(t-t')^3}}dt', \tag{23}$$

Alternatively,

$$\frac{d}{dt}\int_0^t \frac{\Psi(t,t')\,\nu(t')}{\sqrt{2\pi(t-t')}}dt' = \int_0^t \frac{\frac{\partial}{\partial t'}\left((\Psi(t,t') - 2(t-t')\Psi_t(t,t'))\,\nu(t')\right)}{\sqrt{2\pi(t-t')}}dt'. \tag{24}$$

We emphasize that Eq. (21) is easier to use than Eq. (20) in most situations because it does not involve differentiation. However, if the cumulative outflow $G^{>}(t) = \int_0^t g^{>}(t')\,dt'$ is of interest, the latter equation can be more efficient, since it can be rewritten as follows:

$$G^{>}(t) = -\int_0^t \left(\frac{\left(1 + (t-t')\,\Theta^{>2}(t,t')\right)\Xi^{>}(t,t')\,\nu^{>}(t')}{\sqrt{2\pi(t-t')}} + \frac{db^{>}(t')}{dt}\nu^{>}(t')\right)dt'. \tag{25}$$

We can calculate $g^{<}(t)$ and $g^{><}(t)$ by the same token. It is important to understand that both Eqs (21) and (20) can be used in the one-sided case, however, in the case when two boundaries are present, we can *only* use Eq. (21) because this equation allows calculating $g^{>}$ and $g^{<}$ individually while Eq. (20) calculates the difference $g^{>} - g^{<}$.

2.3. *Generalizations*

If the MHP were applicable only to the standard Wiener process, it would be advantageous, if somewhat narrow in scope. Fortunately, it can be applied to a general diffusion satisfying the so-called Cherkasov's condition, which guarantees that it can be transformed into the standard Wiener process. Such diffusions are studied in Cherkasov (1957), Ricciardi (1976), and Bluman (1980). The applications of Cherkasov's condition in financial mathematics are discussed Sec. 4.2 of Lipton (2001) and Chap. 9 of Lipton (2018).

Consider a diffusion governed by

$$d\tilde{x}_{\tilde{t}} = \delta(\tilde{t}, \tilde{x}_{\tilde{t}})d\tilde{t} + \sigma(\tilde{t}, \tilde{x}_{\tilde{t}})dW_{\tilde{t}}, \quad \tilde{x}_0 = \tilde{z}. \tag{26}$$

We wish to calculate boundary-related quantities, such as the distribution of the hitting time of a given time-dependent barrier $b(\tilde{t})$:

$$\tilde{s} = \inf\{\tilde{t} : \tilde{x}_{\tilde{t}} = \tilde{b}(\tilde{t})\}, \quad \tilde{z} \neq \tilde{b}(0). \tag{27}$$

To this end, we introduce

$$\begin{aligned} \beta(\tilde{t}, \tilde{x}) &= \sigma(\tilde{t}, \tilde{x}) \int^{\tilde{x}} \frac{1}{\sigma(\tilde{t}, y)} dy, \\ \gamma(\tilde{t}, \tilde{x}) &= 2\delta(\tilde{t}, \tilde{x}) - \sigma(\tilde{t}, \tilde{x})\sigma_{\tilde{x}}(\tilde{t}, \tilde{x}) - 2\sigma(\tilde{t}, \tilde{x}) \int^{\tilde{x}} \frac{\sigma_{\tilde{t}}(\tilde{t}, y)}{\sigma^2(\tilde{t}, y)} dy, \end{aligned} \tag{28}$$

where the lower limit of integration is chosen as convenient. Define

$$\begin{aligned} P(\tilde{t}, \tilde{x}) &= \begin{vmatrix} \beta(\tilde{t}, \tilde{x}) & \gamma(\tilde{t}, \tilde{x}) \\ \beta_{\tilde{x}}(\tilde{t}, \tilde{x}) & \gamma_{\tilde{x}}(\tilde{t}, \tilde{x}) \end{vmatrix}, \\ Q(\tilde{t}, \tilde{x}) &= \begin{vmatrix} \sigma(\tilde{t}, \tilde{x}) & \gamma(\tilde{t}, \tilde{x}) \\ \sigma_{\tilde{x}}(\tilde{t}, \tilde{x}) & \gamma_{\tilde{x}}(\tilde{t}, \tilde{x}) \end{vmatrix}, \\ R(\tilde{t}, \tilde{x}) &= \begin{vmatrix} \sigma(\tilde{t}, \tilde{x}) & \beta(\tilde{t}, \tilde{x}) & \gamma(\tilde{t}, \tilde{x}) \\ \sigma_{\tilde{x}}(\tilde{t}, \tilde{x}) & \beta_{\tilde{x}}(\tilde{t}, \tilde{x}) & \gamma_{\tilde{x}}(\tilde{t}, \tilde{x}) \\ \sigma_{\tilde{x}\tilde{x}}(\tilde{t}, \tilde{x}) & \beta_{\tilde{x}\tilde{x}}(\tilde{t}, \tilde{x}) & \gamma_{\tilde{x}\tilde{x}}(\tilde{t}, \tilde{x}) \end{vmatrix}, \end{aligned} \tag{29}$$

and assume that Cherkasov's condition is satisfied, so that

$$R(\tilde{t}, \tilde{x}) \equiv 0. \tag{30}$$

Then we can transform $\tilde{x}$ into the standard Wiener process via the following mapping:

$$\begin{aligned} t &= t(\tilde{t}, \tilde{x}) = \int_0^{\tilde{t}} \Phi^2(u, \tilde{x}) du, \\ x &= x(\tilde{t}, \tilde{x}) = \Phi(\tilde{t}, \tilde{x}) \frac{\beta(\tilde{t}, \tilde{x})}{\sigma(\tilde{t}, \tilde{x})} + \frac{1}{2} \int_0^{\tilde{t}} \Phi(u, \tilde{x}) \frac{P(u, \tilde{x})}{\sigma(u, \tilde{x})} du, \end{aligned} \tag{31}$$

where

$$\Phi(\tilde{t}, \tilde{x}) = \exp\left[-\frac{1}{2} \int_0^t \frac{Q(u, \tilde{x})}{\sigma(u, \tilde{x})} du\right]. \tag{32}$$

In particular, the initial condition becomes

$$z = \frac{\beta(0, \tilde{z})}{\sigma(0, \tilde{z})}. \tag{33}$$

The corresponding transition probability density transforms as follows:

$$\tilde{p}(\tilde{t}, \tilde{x}; \tilde{z}) = \left| \frac{\partial x(\tilde{t}, \tilde{x})}{\partial \tilde{x}} \right| p(t, x; z). \tag{34}$$

Moreover, the boundary transforms to

$$\tilde{b}(\tilde{t}) \to b(t) = \Phi(\tilde{t}, \tilde{b}(\tilde{t})) \frac{\beta(\tilde{t}, \tilde{b}(\tilde{t}))}{\sigma(\tilde{t}, \tilde{b}(\tilde{t}))} + \frac{1}{2} \int^{\tilde{t}} \Phi(u, \tilde{b}(\tilde{t})) \frac{P(u, \tilde{b}(\tilde{t}))}{\sigma(u, \tilde{b}(\tilde{t}))} du. \tag{35}$$

Since the MHP is specifically designed for dealing with curvilinear boundaries, we get a solvable problem. A powerful application of the above approach is demonstrated in Sec. 5, where the hitting time probability distribution for an Ornstein–Uhlenbeck process is studied.

2.4. *Numerics*

There are numerous well-known approaches to solving Volterra equations; see the work of Linz (1985), among many others. We choose the most straightforward approach and show how to solve the following archetypal Volterra equation with weak singularity numerically:

$$\nu(t) + \int_0^t \frac{K(t, t')}{\sqrt{t - t'}} \nu(t')\, dt' = f(t), \tag{36}$$

where $K(t, t')$ is a nonsingular kernel. We write

$$\int_0^t \frac{K(t, t')\nu(t')}{\sqrt{t - t'}} dt' = -2 \int_0^t K(t, t')\nu(t') d\sqrt{t - t'}. \tag{37}$$

We wish to map this equation to a grid $0 = t_0 < t_1 < \cdots < t_N = T$. To this end, we introduce the following notation:

$$f_k = f(t_k), \quad \nu_k = \nu(t_k), \quad K_{k,l} = K(t_k, t_l), \quad \Delta_{k,l} = t_k - t_l. \tag{38}$$

Then, the right-hand side of Eq. (37) can be approximated by the trapezoidal rule as

$$f_k = \nu_k + \sum_{l=1}^{k} (K_{k,l}\nu_l + K_{k,l-1}\nu_{l-1})\Pi_{k,l} = 0, \tag{39}$$

where

$$\Pi_{k,l} = \frac{\Delta_{l,l-1}}{(\sqrt{\Delta_{k,l-1}} + \sqrt{\Delta_{k,l}})}, \tag{40}$$

so that

$$\nu_k = \frac{\left(f_k - K_{k,k-1}\nu_{k-1} - \sum_{l=1}^{k-1}(K_{k,l}\nu_l + K_{k,l-1}\nu_{l-1})\Pi_{k,l}\right)}{(1 + K_{k,k}\sqrt{\Delta_{k,k-1}})}. \tag{41}$$

Thus, ν_k can be found by induction starting with $\nu_0 = f_0$.

Equation (41) is the blueprint for all the subsequent numerical calculations.

3. The Structural Default Model

3.1. *Preliminaries*

The original, and straightforward, structural default model was introduced by Merton, Merton (1974), who assumed that default could happen only at debt maturity. His model was extended by Black & Cox (1976) who considered the default, which can happen at any time by introducing flat default boundary representing debt covenants. Numerous authors expanded their model including Hyer *et al.* (1998), Hull & White (2001), and Avellaneda & Zhu (2001), who considered a curvilinear boundary whose shape can be calibrated to the market default probability. One of the major unsolved issues with the above model was articulated by Hyer *et al.* (1998), who pointed out that, unless the shape of the default boundary is very carefully chosen, the probability of short-term default is too low. This issue was addressed by several authors, including Finkelstein & Lardy (2001), Hilberink & Rogers (2002), and Lipton (2002), who proposed to introduce jumps and/or uncertainty to increase this probability. We show below that it is possible to calibrate the default boundary in such a way that constant default intensity can be matched. We emphasize that the direct problem — calculating the default probability given the boundary — is linear (albeit relatively involved), while the inverse problem — finding the boundary given the default probability — is nonlinear (and hence even more involved). Additional details are given in Lipton & Kaushansky (2020b).

3.2. *Formulation*

We wish to find the boundary for a structural default model corresponding to a constant default intensity η. We denote the corresponding default probability by

$$\pi(t) = 1 - e^{-\eta t}. \tag{42}$$

The introduced time τ is such that the default is impossible for $t < \tau$. Thus the default boundary starts at $t = \tau$. The idea is to calculate the corresponding boundary $b(t; \tau, \eta)$, provided it exists, and then let $\tau \to 0$.

It is clear that at time $t = \tau - 0$, the transition probability is

$$p(\tau, x) = H(\tau, x), \tag{43}$$

where H is the heat kernel:

$$H(\tau, x) = \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}}. \tag{44}$$

At time $t = \tau$, the first possibility of default occurs. For $t > \tau$, the transition probability satisfies the following Fokker–Planck problem:

$$\begin{aligned} &\frac{\partial}{\partial t}p(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x), \quad b(t) \le x < \infty, \\ &p(\tau, x) = H(\tau, x), \quad p(t, b(t)) = 0, \quad p(t, x \to \infty) \to 0. \end{aligned} \tag{45}$$

The default probability density $g(t)$ is given by

$$g(t) = \frac{1}{2}\frac{\partial}{\partial x}p(t, b(t)). \tag{46}$$

Alternatively,

$$\pi(t) = 1 - \int_{b(t)}^{\infty} p(t,x)dx, \quad g(t) = \frac{d\pi(t)}{dt}. \tag{47}$$

3.3. *Governing system of integral equations*

We split p as follows:

$$p(t,x) = q(t,x) + r(t,x), \tag{48}$$

where

$$\begin{aligned} &\frac{\partial}{\partial t}q(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}q(t,x), \quad -\infty < x < \infty, \\ &q(\tau, x) = H(\tau,x)\Theta(x - b(\tau)), \quad q(t, x \to -\infty) \to 0, \ q(t, x \to \infty) \to 0, \end{aligned} \tag{49}$$

$$\begin{aligned} &\frac{\partial}{\partial t}r(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}r(t,x), \quad b(t) \le x < \infty, \\ &r(\tau, x) = 0, \quad r(t, b(t)) = -q(t, b(t)), \quad r(t, x \to \infty) \to 0, \end{aligned} \tag{50}$$

and $\Theta(x)$ is the Heaviside function. Solving Eq. (49) as a convolution of heat kernel with the initial condition, we get

$$q(t,x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} N\left(\frac{\frac{ux}{t} - b(\tau)}{\sqrt{u}}\right), \tag{51}$$

where $u = (t-\tau)\tau/t$; see Lipton & Kaushansky (2020b). Thus

$$g(t) = \frac{1}{2}\frac{\partial}{\partial x}r(t,b(t)) - \frac{H(t,b(t))}{2t}\left(b(t)N\left(\frac{\frac{ub(t)}{t}-b(\tau)}{\sqrt{u}}\right) - uH\left(u,\frac{ub(t)}{t}-b(\tau)\right)\right). \tag{52}$$

Accordingly, in view of the discussion in Sec. 2.2, we need to solve the following system of integral equations:

$$\begin{aligned}
&\nu(t) + \int_\tau^t \frac{\Theta(t,t')\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')}}dt' + H(t,b(t))N\left(\frac{ub(t)-tb(\tau)}{t\sqrt{u}}\right) = 0,\\
&\eta e^{-\eta t} + \left(\frac{1}{\sqrt{2\pi t}} + \frac{db(t)}{dt}\right)\nu(t) + \frac{1}{2}\int_\tau^t \frac{\Phi(t,t') + \Theta^2(t,t')\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')}}dt'\\
&\quad + \frac{H(t,b(t))}{2t}\left(b(t)N\left(\frac{\frac{ub(t)}{t}-b(\tau)}{\sqrt{u}}\right) - uH\left(u,\frac{ub(t)}{t}-b(\tau)\right)\right) = 0.
\end{aligned} \tag{53}$$

Alternatively, we can rewrite Eqs. (53) in integrated form as follows:

$$\begin{aligned}
&\nu(t) + \int_\tau^t \frac{\Theta(t,t')\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')}}dt' + H(t,b(t))N\left(\frac{ub(t)-tb(\tau)}{t\sqrt{u}}\right) = 0,\\
&1 - e^{-\eta t} + \int_\tau^t \frac{\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')}}dt' - N\left(\frac{b(t)}{\sqrt{t}}\right) - N\left(\frac{\sqrt{t}b(\tau)}{\sqrt{u(u+t)}}\right)\\
&\quad + \mathrm{BVN}\left(\frac{\sqrt{t}b(\tau)}{\sqrt{u(u+t)}}, \frac{b(t)}{\sqrt{t}}; \sqrt{\frac{u}{u+t}}\right) = 0,
\end{aligned} \tag{54}$$

where $\mathrm{BVN}(\cdot,\cdot;\cdot)$ is the bivariate normal distribution.

We postpone the discussion of the corresponding numerics until the next section, where a more general case is considered.

3.4. *The choice of b_τ*

Recall that the default probability has the form

$$\pi(t) = 1 - e^{-\eta t}. \tag{55}$$

The barrier has to start at $\tau = \hat{\tau}$, $\hat{\tau} \to 0$, and there should be no barrier before that. We wish to find $b(\hat{\tau})$ such that

$$\pi(\hat{\tau}) = 1 - \int_{b(\hat{\tau})}^\infty \frac{\exp\left(-\frac{x^2}{2\hat{\tau}}\right)}{\sqrt{2\pi\hat{\tau}}}dx = 1 - N\left(-\frac{b(\hat{\tau})}{\sqrt{\hat{\tau}}}\right) = 1 - e^{-\eta\hat{\tau}}. \tag{56}$$

Thus,

$$N\left(-\frac{b(\hat{\tau})}{\sqrt{\hat{\tau}}}\right) = e^{-\eta\hat{\tau}}, \tag{57}$$

and

$$b(\hat{\tau}) = -\sqrt{\hat{\tau}}N^{-1}(e^{-\eta\hat{\tau}}). \tag{58}$$

Now,

$$N^{-1}(y) \underset{y\to 1}{\sim} \sqrt{2f(\eta)}, \tag{59}$$

where

$$\eta = -\ln(2\sqrt{\pi}(1-y)), \quad f(\eta) = \eta - \frac{\ln\eta}{2} + \frac{\ln\eta - 2}{4\eta} + \frac{(\ln\eta)^2 - 6\ln\eta + 14}{16\eta^2}, \tag{60}$$

so that

$$b(\hat{\tau}) = -\sqrt{2\hat{\tau}f(-\ln(2\sqrt{\pi}(1-e^{-\eta\hat{\tau}})))} \approx -\sqrt{2\hat{\tau}\ln\left(\frac{1}{2\sqrt{\pi}\eta\hat{\tau}}\right)}. \tag{61}$$

3.5. *Default boundaries*

Default boundaries calibrated to several representative values of η are shown in Fig. 1.

We show that solutions of Eqs. (53) and (54) coincide modulo numerical error in Fig. 2.

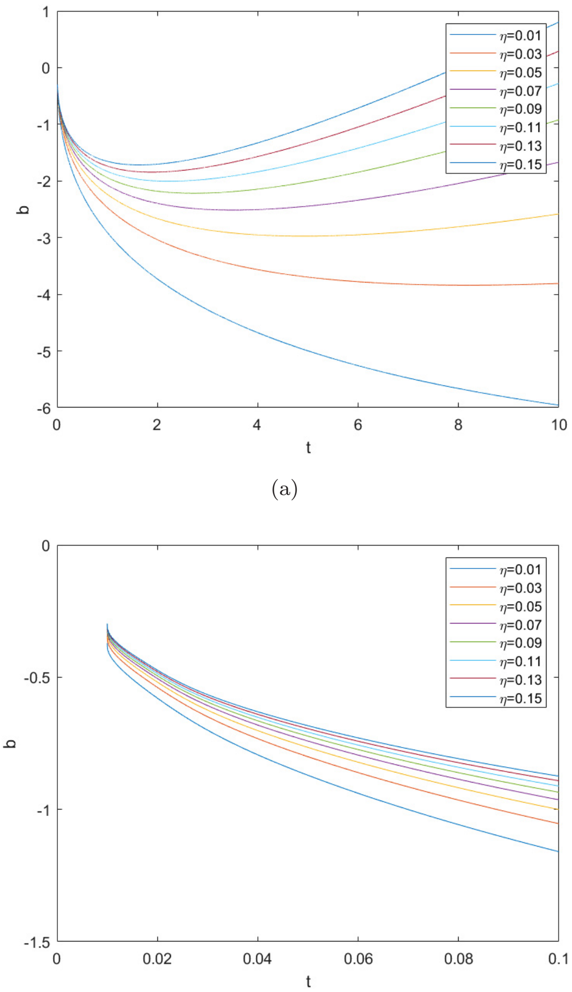
3.6. *Main conjecture*

Conjecture. *For a given time interval* $I^{(T)} = [0, T]$, *there exists a parameter interval* $I^{(\eta)}(T) = [0, \eta^*(T)]$ *such that for any* $\eta \in I^{(\eta)}(T)$, *the default boundary* $b(t)$ *can be calibrated to the default intensity* η. *We can construct the corresponding boundary as follows*:

$$b(t;\eta) = \lim_{\tau\to 0} b(t;\tau,\eta), \quad 0 < t \le T, \tag{62}$$

where $b(t;\tau,\eta)$ *is found by solving either Eqs.* (53) *or* (54).

We illustrate our conjecture in Fig. 3.



(a)

(b)

Fig. 1. In panels (a) and (b), we show the default boundaries for several representative values of the default intensity η. We choose $\tau = 0.01$. In panel (a), we choose $0.01 < t < 10.0$ to capture their overall behavior; in panel (b), we choose $0.01 < t < 0.1$ so that small features can be shown.

4. Mean-Field Banking System

4.1. *Preliminaries*

No bank is an island — they operate as a group. Tangible links between banks manifest themselves via interbank loans; intangible links are manifold — overall sentiment, ease of doing business, and others. Hence, to build a meaningful structural default model for a bank, one needs to take into account this bank's interactions with all the banks whom it lends to or borrows from. Eisenberg and Noe developed a Merton-like model of the bank default (default can happen only at

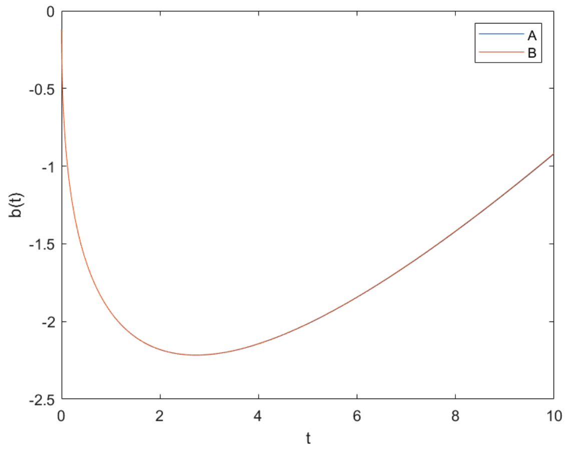
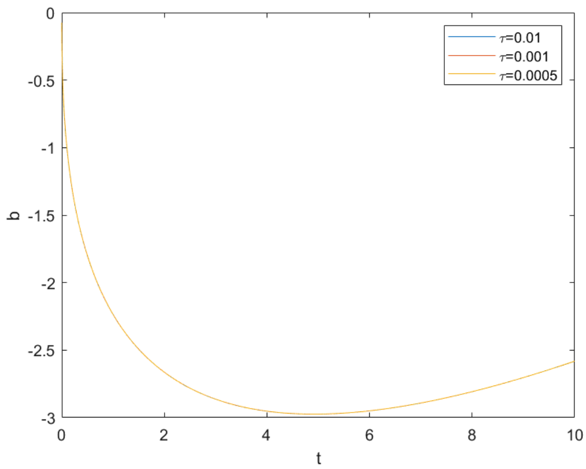


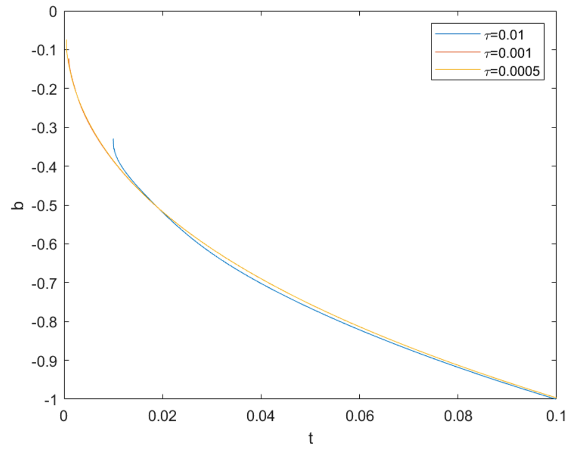
Fig. 2. Here we choose the default intensity $\eta = 0.09$ and show that boundaries calculated by solving Eqs. (53) and (54) coincide modulo numerical errors.

maturity) in a seminal paper Eisenberg & Noe (2001). The present author extended the Eisenberg–Noe model to the Black–Cox setting (default can happen at any time before maturity provided that debt covenants are violated); see Lipton (2016). Lipton's work was subsequently generalized in Itkin & Lipton (2015) and Itkin & Lipton (2017). Recently, several authors considered the interconnected banking system in the mean-field framework and studied a representative bank; see Hambly

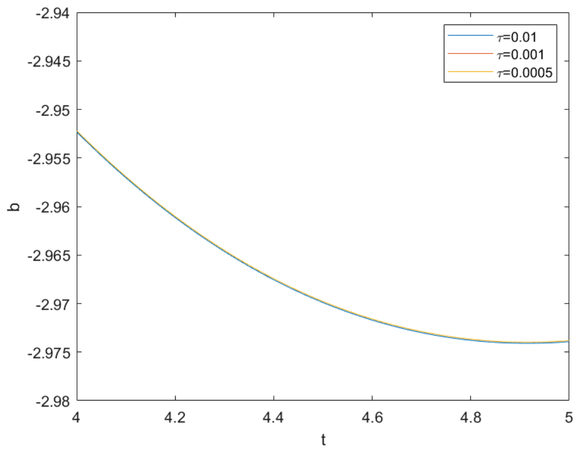


(a)

Fig. 3. Here we choose the default intensity $\eta = 0.05$ and illustrate our main conjecture numerically by constructing three boundaries corresponding to $\tau = 0.01$, 0.001, and 0.0005, respectively. It is clear that after a short initial period, these boundaries begin to overlap.



(b)



(c)

Fig. 3. (*Continued*)

et al. (2018), Ichiba *et al.* (2018), Kaushansky & Reisinger (2019), and Nadtochiy & Shkolnikov (2017, 2018) among many others. In this section, we also use the mean-field approach. Additional details are given in Lipton *et al.* (2019).

4.2. *Interconnected banking system*

We follow Lipton (2016) and assume that the dynamics of bank i's total external assets is governed by

$$\frac{dA_t^i}{A_t^i} = \mu_i \, dt + \sigma_i \, dW_t^i, \tag{63}$$

where W^i are independent standard Brownian motions for $1 \leq i \leq n$, and the liabilities, both external L_i and mutual L_{ij}, are constant.

Bank i is assumed to default when its assets fall below a certain threshold determined by its liabilities, namely at time $\tau_i = \inf\{t : A_t^i \leq \Lambda_t^i\}$, where Λ^i is a default boundary which we now work out. At time $t = 0$,

$$\Lambda_0^i = R_i \left(L_i + \sum_{j \neq i} L_{ij} \right) - \sum_{j \neq i} L_{ji}, \tag{64}$$

where R_i is the recovery rate of bank i. If bank k defaults at time t, the default boundary of bank i jumps by $\Delta\Lambda_t^i = (1 - R_i R_k) L_{ki}$.

The distance to default $Y_t^i = \log(A_t^i/\Lambda_t^i)/\sigma$ has the following dynamics:

$$Y_t^i = Y_0^i + \left(\frac{\mu - \sigma^2}{2} \right) t + W_t^i - \frac{1}{\sigma} \log \left(1 + \frac{\gamma}{N} \sum_{k \neq i} (1 - R^2) \frac{1}{\Lambda_0} \mathbf{1}_{\{\tau_k \leq t\}} \right), \tag{65}$$

or, approximately,

$$Y_t^i = Y_t^0 + \left(\frac{\mu - \sigma^2}{2} \right) t + W_t^i - \frac{\gamma(1 - R^2)}{\sigma \Lambda_0} L_t^N, \tag{66}$$

where

$$L_t^N = \frac{1}{N} \sum_k \mathbf{1}_{\{\tau_k \leq t\}}. \tag{67}$$

In the limit for $N \to \infty$, all Y_t^i have the same dynamics:

$$\begin{aligned} Y_t &= Y_0 + W_t - \alpha L_t, \\ L_t &= \mathbb{P}(\tau \leq t), \quad \tau = \inf\{t \in [0, T] : Y_t \leq 0\}, \end{aligned} \tag{68}$$

where $\alpha = \gamma(1 - R^2)/\sigma\Lambda_0$ characterizes the strength of interbank interactions. Thus, we are dealing with a mean-field problem — the behavior of a representative bank depends on the behavior of all other banks, and all of them have the same dynamics. Hence, the problem in question is nonlinear.

We follow Lipton *et al.* (2019) and write the increasing process L as

$$\alpha L_t = -\int_0^t \mu(t')\, dt' = -M(t), \tag{69}$$

for some negative μ, so that p satisfies

$$\begin{aligned} &\frac{\partial}{\partial t} p(t, x; z) = -\mu(t) \frac{\partial}{\partial x} p(t, x; z) + \frac{1}{2} \frac{\partial^2}{\partial x^2} p(t, x; z), \quad 0 \leq x < \infty, \\ &p(0, x; z) = \delta_z(x), \quad p(t, 0; z) = 0, \quad p(t, x \to \infty) \to 0. \end{aligned} \tag{70}$$

As we already know,

$$g(t; z) \equiv \frac{dL_t}{dt} = \frac{1}{2} p_x(t, 0; z), \tag{71}$$

so that Eqs. (70) can be written in the self-consistent form

$$\begin{aligned}&\frac{\partial}{\partial t}p(t,x;z)=\frac{\alpha}{2}\frac{\partial}{\partial x}p(t,0;z)\frac{\partial}{\partial x}p(t,x;z)+\frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x;z),\quad 0\le x<\infty,\\&p(0,x;z)=\delta_z(x),\quad p(t,0;z)=0,\quad p(t,x\to\infty)\to 0.\end{aligned}\tag{72}$$

The change of variables $(t,x)\to(t,y)=(t,x-M(t))$ yields the familiar initial–boundary-value problem (IBVP):

$$\begin{aligned}&\frac{\partial}{\partial t}p(t,y)=\frac{1}{2}p_{yy}(t,y),\quad 0\le y<\infty,\\&p(0,y)=\delta_z(y),\quad p(t,-M(t))=0,\quad p(t,y\to\infty)\to 0.\end{aligned}\tag{73}$$

As before, we split p in two parts,

$$p(t,y)=H(t,y)+r(t,y),\tag{74}$$

where $H(t,y)$ is the standard heat kernel, while r is the solution of the following problem:

$$\begin{aligned}&\frac{\partial}{\partial t}r(t,y)=\frac{1}{2}\frac{\partial^2}{\partial y^2}r(t,y),\quad 0\le y<\infty,\\&r(0,y)=0,\quad r(t,-M(t))=-\frac{\exp\left(-\frac{(M(t)+z)^2}{2t}\right)}{\sqrt{2\pi t}},\quad r(t,y\to\infty)\to 0.\end{aligned}\tag{75}$$

4.3. *Governing system of integral equations*

Using our standard approach, we obtain the following system of nonlinear Volterra integral equations:

$$\begin{aligned}&\nu(t)+\int_0^t\frac{\Theta(t,t')\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')}}dt'+H(t,t\Theta(t,0)-z)=0,\\&\mu(t)+\left(\frac{1}{\sqrt{2\pi t}}+\alpha\mu(t)\right)\nu(t)\\&\quad+\frac{1}{2}\int_0^t\frac{\Phi(t,t')+\Theta^2(t,t')\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')}}dt'+\frac{(t\Theta(t,0)-z)H(t,t\Theta(t,0)-z)}{2t}=0,\end{aligned}\tag{76}$$

where

$$\begin{aligned}&\Theta(t,t')=\frac{\alpha\int_{t'}^t\mu(t'')dt''}{(t-t')},\quad \Xi(t,t')=e^{-\frac{(t-t')\Theta^2(t,t')}{2}},\\&\Phi(t,t')=\frac{(\nu(t)-\Xi(t,t')\nu(t'))}{(t-t')},\\&\Theta(t,t)=\alpha\mu(t),\quad \Xi(t,t)=1,\quad \Phi(t,t)=\nu'(t)+\frac{1}{2}\alpha^2\mu^2(t)\nu(t).\end{aligned}$$

4.4. *Numerical solution*

In the spirit of Eq. (39), we get the following approximation for Eqs. (76) for $k > 0$:

$$
\begin{aligned}
&\nu_k + \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{k} (P^{(1)}_{k,l} \nu_l + P^{(1)}_{k,l-1} \nu_{l-1}) \Pi_{k,l} + \vartheta_k = 0, \\
&\mu_k + \left(\frac{1}{\sqrt{2\pi \Delta_{k,0}}} + \alpha \mu_k \right) \nu_k + \frac{1}{2\sqrt{2\pi}} \sum_{l=1}^{k} \\
&\quad \times (\Phi_{k,l} + \Phi_{k,l-1} + P^{(2)}_{k,l} \nu_l + P^{(2)}_{k,l-1} \nu_{l-1}) \Pi_{k,l} + \iota_k = 0.
\end{aligned}
\tag{77}
$$

Here and below we use the following notation:

$$
\begin{aligned}
&\Theta_{k,l} = \alpha \frac{\sum_{i=l+1}^{k} (\mu_i + \mu_{i-1}) \Delta_{i,i-1}}{2\Delta_{k,l}}, \quad P^{(i)}_{k,l} = \Theta^{i}_{k,l} e^{-\frac{\Delta_{k,l} \Theta^2_{k,l}}{2}}, \\
&Q_{k,l} = P^{(2)}_{k,l} - \frac{P^{(0)}_{k,l}}{\Delta_{k,l}}, \quad \Phi_{k,l} = \frac{\nu_k - P^{(0)}_{k,l} \nu_{k-1}}{\Delta_{k,l}}, \quad k > l, \\
&\Theta_{k,k} = \alpha \mu_k, \quad P^{(i)}_{k,k} = \alpha^i \mu^i_k, \quad Q_{k,k} \quad \text{undefined}, \\
&\Phi_{k,k} = \frac{\nu_k - \nu_{k-1}}{\Delta_{k,k-1}} + \frac{1}{2} \alpha^2 \mu^2_k \nu_k, \\
&\vartheta_k = H(\Delta_{k,0}, \Delta_{k,0} \Theta_{k,0} - z), \quad \iota_k = \frac{(\Delta_{k,0} \Theta_{k,0} - z) \vartheta_k}{2\Delta_{k,0}}, \quad k > 0.
\end{aligned}
\tag{78}
$$

For $k = 0$, we have

$$(\nu_0, \mu_0) = (0, 0).$$

For $k = 1$, we have

$$
\begin{aligned}
&\nu_1 = -\frac{H\left(\Delta_{1,0}, \frac{\Delta_{1,0} \alpha \mu_1}{2} - z \right)}{\left(1 + \sqrt{\frac{\Delta_{1,0}}{2\pi}} \alpha \mu_1 \right)}, \\
&\mu_1 - \left(\frac{\left(\frac{1}{\sqrt{2\pi \Delta_{1,0}}} + \alpha \mu_1 + \frac{\alpha^2 \mu_1^2}{2\sqrt{2\pi \Delta_{1,0}}} \right)}{\left(1 + \sqrt{\frac{\Delta_{1,0}}{2\pi}} \alpha \mu_1 \right)} - \frac{\left(\frac{\Delta_{1,0} \alpha \mu_1}{2} - z \right)}{2\Delta_{1,0}} \right) H \\
&\quad \times \left(\Delta_{1,0}, \frac{\Delta_{1,0} \alpha \mu_1}{2} - z \right) = 0,
\end{aligned}
\tag{79}
$$

where the nonlinear equation for μ_1 has to be solved by the Newton–Raphson method.

For $k > 1$, we have

$$\begin{aligned}
&\left(1+\sqrt{\frac{\Delta_{k,k-1}}{2\pi}}\alpha\mu_k\right)\nu_k+\sqrt{\frac{\Delta_{k,k-1}}{2\pi}}P^{(1)}_{k,k-1}\nu_{k-1}\\
&\quad+\frac{1}{\sqrt{2\pi}}\sum_{l=1}^{k-1}(P^{(1)}_{k,l}\nu_l+P^{(1)}_{k,l-1}\nu_{l-1})\Pi_{k,l}+\vartheta_k=0,\\
&\mu_k+\left(\frac{1}{\sqrt{2\pi\Delta_{k,0}}}+\alpha\mu_k+\frac{\alpha^2\mu_k^2}{2\sqrt{2\pi\Delta_{k,k-1}}}+\frac{1}{2\sqrt{2\pi}}\sum_{l=1}^{k-1}\frac{(\Delta_{k,l}+\Delta_{k,l-1})\Pi_{k,l}}{\Delta_{k,l}\Delta_{k,l-1}}\right)\nu_k\\
&\quad+\frac{1}{2}\sqrt{\frac{\Delta_{k,k-1}}{2\pi}}Q_{k,k-1}\nu_{k-1}+\frac{1}{2\sqrt{2\pi}}\sum_{l=1}^{k-1}(Q_{k,l}\nu_l+Q_{k,l-1}\nu_{l-1})\Pi_{k,l}+\iota_k=0.
\end{aligned} \tag{80}$$

Assuming that $(\nu_1,\mu_1),\ldots,(\nu_{k-1},\mu_{k-1})$ are known, we can express ν_k in terms of μ_k:

$$\nu_k=-\frac{\left(\sqrt{\frac{\Delta_{k,k-1}}{2\pi}}P^{(1)}_{k,k-1}\nu_{k-1}+\frac{1}{\sqrt{2\pi}}\sum_{l=1}^{k-1}(P^{(1)}_{k,l}\nu_l+P^{(1)}_{k,l-1}\nu_{l-1})\Pi_{k,l}+\vartheta_k\right)}{\left(1+\sqrt{\frac{\Delta_{k,k-1}}{2\pi}}\alpha\mu_k\right)}, \tag{81}$$

and obtain a nonlinear equation for μ_k:

$$\begin{aligned}
&\mu_k-\frac{\left(\frac{1}{\sqrt{2\pi\Delta_{k,0}}}+\alpha\mu_k+\frac{\alpha^2\mu_k^2}{2\sqrt{2\pi\Delta_{k,k-1}}}+\frac{1}{2\sqrt{2\pi}}\sum_{l=1}^{k-1}\frac{(\Delta_{k,l}+\Delta_{k,l-1})\Pi_{k,l}}{\Delta_{k,l}\Delta_{k,l-1}}\right)}{\left(1+\sqrt{\frac{\Delta_{k,k-1}}{2\pi}}\alpha\mu_k\right)}\\
&\quad\times\left(\sqrt{\frac{\Delta_{k,k-1}}{2\pi}}P^{(1)}_{k,k-1}\nu_{k-1}+\frac{1}{\sqrt{2\pi}}\sum_{l=1}^{k-1}(P^{(1)}_{k,l}\nu_l+P^{(1)}_{k,l-1}\nu_{l-1})\Pi_{k,l}+\vartheta_k\right)\\
&\quad+\frac{(Q_{k,k-1}\Delta_{k,k-1}-1)\nu_{k-1}}{2\sqrt{2\pi\Delta_{k,k-1}}}+\frac{1}{2\sqrt{2\pi}}\sum_{l=1}^{k-1}(Q_{k,l}\nu_l+Q_{k,l-1}\nu_{l-1})\Pi_{k,l}+\iota_k=0,
\end{aligned} \tag{82}$$

which again is solved by the Newton–Raphson method.

In Fig. 4, we show the cumulative loss probability for several representative values of α.

A striking feature of this figure is the "phase transition" occurring at $\alpha \approx 1.0$, when the default after a finite time becomes inevitable. By contrast, for $\alpha = 0$, the default probability reaches unity only asymptotically when $t \to \infty$.

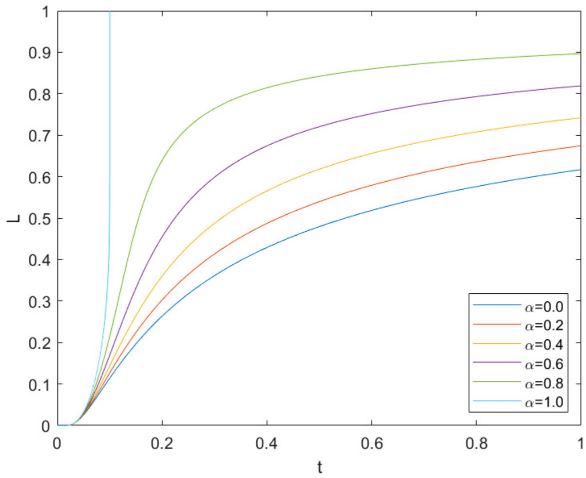
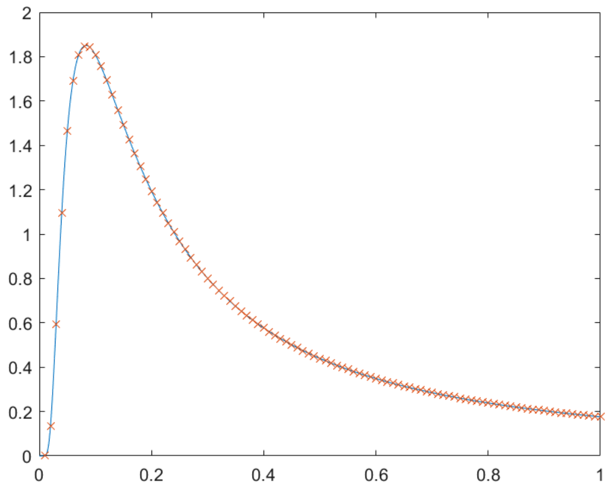


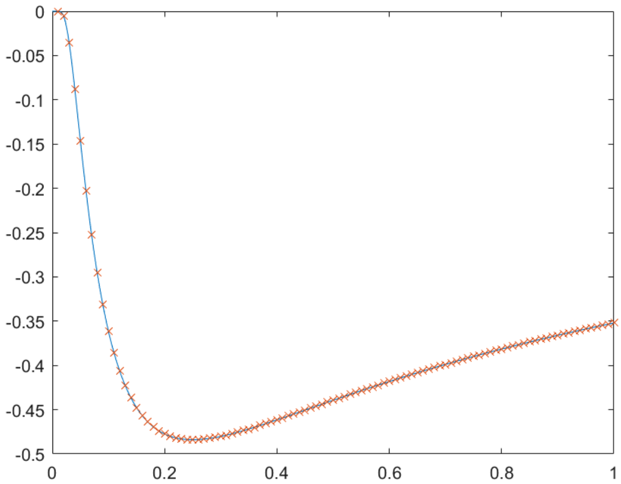
Fig. 4. Here we demonstrate the loss probability for the initial position $z = 0.5$ and several representative values of α, which characterizes the strength of interbank interactions.

We notice that for $\alpha = 0$, $\mu(t)$, $\nu(t)$ can be calculated analytically. For benchmarking purposes, we compare numerical and analytical results in Figs. 5(a) and 5(b). As usual, the efficiency of the Newton–Raphson method, which is illustrated in Fig. 5(c), is nothing short of miraculous.

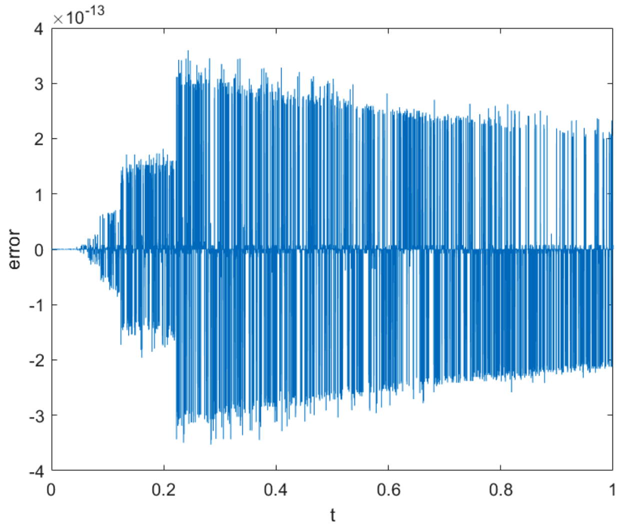


(a)

Fig. 5. In panels (a) and (b), we choose $z = 0.5$, $\alpha = 0$, and we show $\mu(t)$ and $\nu(t)$ calculated numerically and analytically. In panel (c), we choose $z = 0.5$, $\alpha = 0.6$, and we show the error generated by the Newton–Raphson method.



(b)



(c)

Fig. 5. (*Continued*)

In Fig. 6, we represent shifted probability density surfaces $p(t, x - z; z)$ for representative values of α used in Fig. 4.

The shift is made in order to make the connection with Sec. 3 more transparent; after this shift all the processes start at zero and the boundaries are given by $b = -0.5$.

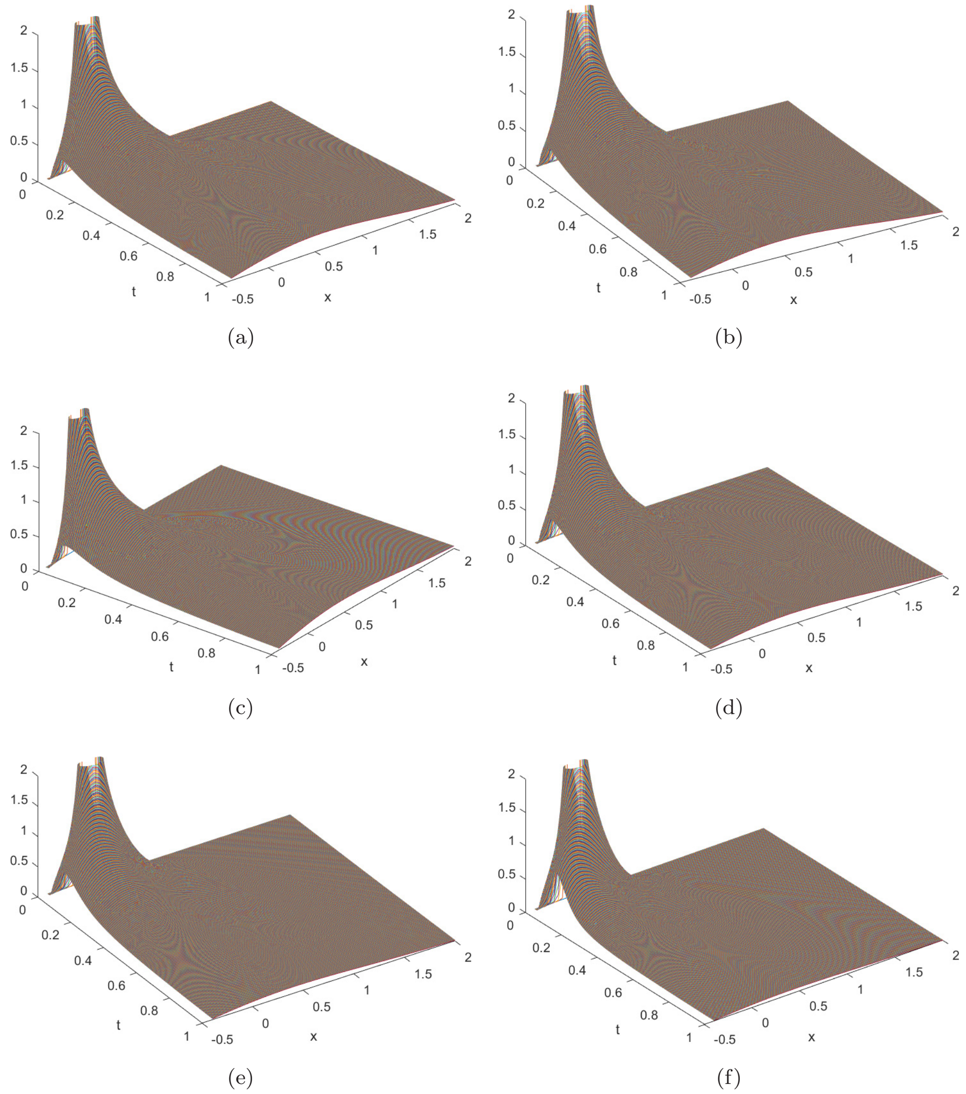


Fig. 6. In panels (a)–(f), we show the probability density function $p(t, x-z; z)$. We shift the domain down by $-z$ in order to make the comparison with the structural default model considered in Sec. 3 more transparent. In panels (a) and (b), we show the analytical and numerical results for $\alpha = 0$. In panels (c)–(f), we show the numerical results for $\alpha = 0.2$, 0.4, 0.6, and 0.8, respectively.

5. Hitting Time Probability Distribution for an Ornstein–Uhlenbeck Process

5.1. *Preliminaries*

In a seminal paper, Fortet (1943) developed an original approach to calculating probability distribution of the hitting time for a diffusion process. Fortet's equation can be viewed as a variant of the Einstein–von Smoluchowski equation (Einstein 1905, von Smoluchowski 1906). A general overview can be found in Borodin & Salminen (2012) and Breiman (1967).

Numerous attempts to find an analytical result for the Ornstein–Uhlenbeck process have been made since 1998 when Leblanc & Scaillet (1998) first derived an analytical formula, which contained a mistake. Two years later, Leblanc *et al.* (2000) published a correction on the paper; unfortunately, the correction was erroneous as well, as was shown by Göing-Jaeschke & Yor (2003).

Several authors used the Laplace transform to find a formal semi-analytical solution (Alili *et al.* 2005, Linetsky 2004, Ricciardi & Sato 1988).

In this section, we use the EMHP to calculate the distribution of the hitting time for an OU process. Our approach is semi-analytical and can handle both constant and time-dependent parameters. It is worth noting that the latter case cannot be solved using the Laplace transform method. Additional information can be found in Lipton & Kaushansky (2020a).

5.2. *Main equations*

To calculate the density $g(t, z)$ of the hitting time probability distribution, we need to solve the following forward problem:

$$\begin{gathered}\frac{\partial}{\partial t}p(t,x;z) = p(t,x;z) + x\frac{\partial}{\partial x}p(t,x;z) + \frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x;z),\\ p(0,x;z) = \delta_z(x), \quad p(t,b(t);z) = 0, \quad p(t,x;z\to\infty)\to 0.\end{gathered} \tag{83}$$

This distribution is given by

$$g(t,z) = \frac{1}{2}\frac{\partial}{\partial x}p(t,b;z). \tag{84}$$

5.3. *Particular case, $b = 0$*

Before solving the general problem via the EMHP, let us consider a particular case of $b = 0$. Green's function for the OU process in question has the form

$$G(t,x;z) = e^t H(\eta(t), e^t x - z), \tag{85}$$

where

$$\eta(t) = \frac{e^{2t}-1}{2} = e^t\sinh(t). \tag{86}$$

Since $b = 0$, the method of images works, so that

$$p(t,x;z) = e^t H(\eta(t), e^t x - z) - e^t H(\eta(t), e^t x + z),$$

$$g(t) = \frac{1}{2}\frac{\partial}{\partial x}p(t,0) = \frac{ze^{2t}H(\eta(t),-z)}{\eta(t)},$$
$$G(t) = \int_0^t g(t')dt' = 2N\left(-\frac{z}{\sqrt{\eta(t)}}\right). \tag{87}$$

This result is useful for benchmarking purposes.

5.4. *General case*

To be concrete, consider the case $z > b(0)$. We wish to transform the IBVP (83) into the standard IBVP for a heat equation with a moving boundary. To this end, we introduce new independent and dependent variables as follows:

$$q(\tau,\xi) = e^{-t}p(t,x), \quad \tau = \eta(t), \quad \xi = e^t x,$$
$$p(t,x) = \sqrt{1+2\tau}q(\tau,\xi), \quad t = \ln(\sqrt{1+2\tau}), \quad x = \frac{\xi}{\sqrt{1+2\tau}}, \tag{88}$$

and get the IBVP of the form

$$\frac{\partial}{\partial\tau}q(\tau,\xi) = \frac{1}{2}\frac{\partial^2}{\partial\xi^2}q(\tau,\xi), \quad \beta(\tau) \le \xi < \infty,$$
$$q(0,\xi) = \delta_z(\xi), \quad q(\tau,\beta(\tau)) = 0, \quad q(\tau,\xi\to\infty) \to 0. \tag{89}$$

Here

$$\beta(\tau) = \sqrt{1+2\tau}\tilde{b}(\ln(\sqrt{1+2\tau})). \tag{90}$$

5.5. *The governing system of integral equations*

The corresponding system of Volterra integral equations has the form

$$\nu(\tau) + \int_0^\tau \frac{\Theta(\tau,\tau')\Xi(\tau,\tau')\nu(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau' + H(\tau,\beta(\tau)-z) = 0,$$
$$\mu(\tau) + \left(\frac{1}{\sqrt{2\pi\tau}} + \beta'(\tau)\right)\nu(\tau) + \frac{1}{2}\int_0^\tau \frac{\Phi(\tau,\tau') + \Theta^2(\tau,\tau')\Xi(\tau,\tau')\nu(\tau')}{\sqrt{2\pi(\tau-\tau')}}d\tau'$$
$$+ \frac{(\beta(\tau)-z)H(\tau,\beta(\tau)-z)}{2\tau} = 0, \tag{91}$$

where

$$\mu(\tau) = (1+2\tau)g(\ln(\sqrt{1+2\tau})). \tag{92}$$

This system is linear, so that $\mu(\tau)$ is expressed in terms of $\nu(\tau)$ directly and there is no need to use the Newton–Raphson method.

5.6. *Flat boundary*

Assuming that the boundary is flat, we can simplify Eqs. (91) somewhat. We notice that

$$\frac{\beta(\tau)-\beta(\tau')}{\tau-\tau'} = \frac{2b}{\sqrt{1+2\tau}+\sqrt{1+2\tau'}}, \tag{93}$$

so we introduce

$$\theta = \sqrt{1+2\tau}-1, \quad \theta' = \sqrt{1+2\tau'}-1, \quad 0 \le \theta' \le \theta < \infty, \tag{94}$$

and write the first equation of (91) in the form

$$\nu(\theta) + \frac{2b}{\sqrt{\pi}} \int_0^\theta \frac{\exp\left(-\frac{b^2(\theta-\theta')}{(2+\theta+\theta')}\right)(1+\theta')\nu(\theta')}{\sqrt{(2+\theta+\theta')^3(\theta-\theta')}} d\theta' + \frac{e^{-\frac{((1+\theta)b-z)^2}{((1+\theta)^2-1)}}}{\sqrt{\pi((1+\theta)^2-1)}} = 0. \tag{95}$$

Provided that $\nu(\theta)$ is known, we can represent $g(t)$ in the form

$$g(t) = -\frac{(e^t b - z)\exp\left(-\frac{(e^t b-z)^2}{(e^{2t}-1)} + 2t\right)}{\sqrt{\pi(e^{2t}-1)^3}} - \left(e^t b + \frac{e^{2t}}{\sqrt{\pi(e^{2t}-1)}}\right)\nu(t) + \frac{1}{\sqrt{\pi}}e^{2t}$$
$$\times \int_0^\theta \frac{\left(\left(1-2b^2\frac{(\theta-\theta')}{(2+\theta+\theta')}\right)\exp\left(-b^2\frac{(\theta-\theta')}{(2+\theta+\theta')}\right)\nu(\theta')-\nu(\theta)\right)(1+\theta')}{\sqrt{(2+\theta+\theta')^3(\theta-\theta')^3}} d\theta'. \tag{96}$$

It is worth noting that the analytical solution is available in two cases: (a) when $b = 0$ the solution can be found by using the method of images and (b) when $b(t) = Ae^{-t} + Be^t$, the boundary transforms into a linear boundary $2B\tau + A + B$, which can be treated by the method of images as well.

We show the probability density function (pdf) and the cumulative density function (cdf) for the hitting time in Fig. 7. It is interesting to note that the undulation of the boundary causes considerable variations in the pdfs, which are naturally less pronounced for the corresponding cdfs.

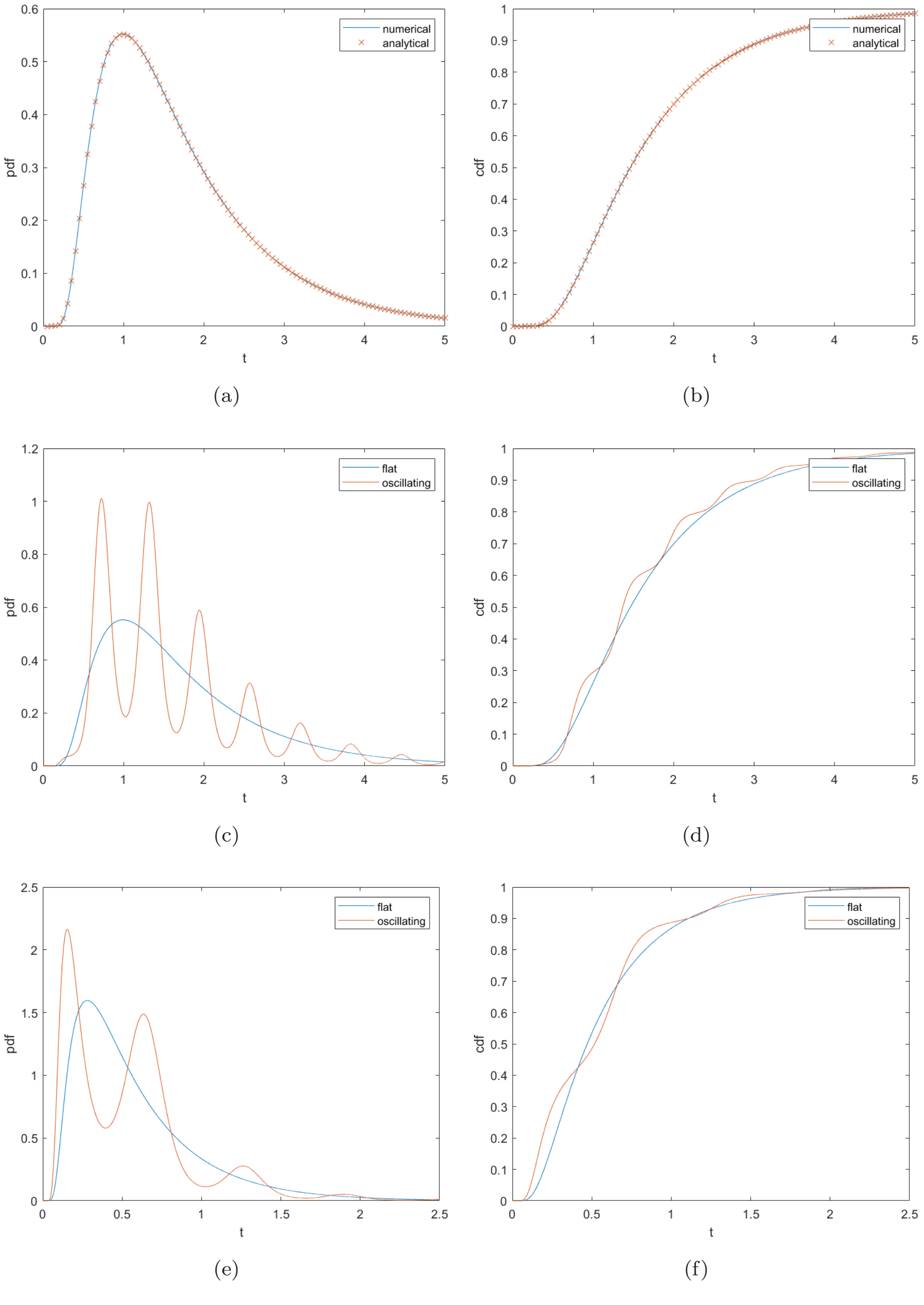


Fig. 7. In panels (a)–(f), we show the pdf and cdf for the hitting time probability distribution. In panels (a) and (b), $z = 2$ and $b(t) = 0$, so that both the numerical and analytical expressions are available. These expressions are in perfect agreement. In panels (c) and (d), $z = 2$, $b(t) = 0$, and $b(t) = 0.2\sin(10.0t)$. In panels (e) and (f), $z = 2$, $b(t) = 1.0$, and $b(t) = 1.0 + 0.2\sin(10.0t)$. Variations in the pdf caused by the barrier undulations are astonishingly profound.

5.7. *Abel integral equation*

Consider Eq. (95), which we got for the standard OU process. For small values of θ, this equation can be approximated by an Abel integral equation of the second kind,

$$\nu(\theta) + \frac{b}{\sqrt{2\pi}} \int_0^\theta \frac{1}{\sqrt{\theta - \theta'}} \nu(\theta') d\theta' + H(\theta, b - z) = 0. \tag{97}$$

This equation can be solved analytically using direct–inverse Laplace transforms. The direct Laplace transform yields

$$\bar{\nu}(\Lambda) + b\frac{\bar{\nu}(\Lambda)}{\sqrt{2\Lambda}} + \frac{e^{-\sqrt{2\Lambda}(z-b)}}{\sqrt{2\Lambda}} = 0. \tag{98}$$

Then, $\bar{\nu}(\Lambda)$ can be expressed as

$$\bar{\nu}(\Lambda) = -\frac{e^{-\sqrt{2\Lambda}(z-b)}}{\sqrt{2\Lambda} + b}. \tag{99}$$

Taking the inverse Laplace transform, we get the final expression for $\nu(\theta)$,

$$\nu(\theta) = be^{\frac{b^2}{2}\theta + b(z-b)} N\left(-\frac{b\theta + z - b}{\sqrt{\theta}}\right) - \frac{\exp\left(-\frac{(b-z)^2}{2\theta}\right)}{\sqrt{2\pi\theta}}. \tag{100}$$

Alternatively, one can represent an analytical solution of an Abel equation,

$$y(t) + \xi \int_0^t \frac{y(s)ds}{\sqrt{t-s}} = f(t), \tag{101}$$

in the form

$$y(t) = F(t) + \pi\xi^2 \int_0^t \exp[\pi\xi^2(t-s)]F(s)ds, \tag{102}$$

where

$$F(t) = f(t) - \xi \int_0^t \frac{f(s)ds}{\sqrt{t-s}}, \tag{103}$$

see Polyanin & Manzhirov (1998).

Abel equations naturally arise in many financial mathematics situations, mainly, when fractional differentiation is involved, see e.g. Andersen & Lipton (2013).

6. The Supercooled Stefan Problem

Here and in Sec. 7, we deal with relatively rare instances when the financial mathematics results can be successfully used in the broader applied mathematics context rather than the other way around.

The Stefan problem is of great theoretical and practical interest, see e.g. (Kamenomostskaja 1961, Rubinstein 1971, Delarue *et al.* 2019) and references therein. The classical Stefan problem studies the evolving boundary between the two phases of the same medium, such as ice and water. Thus, this problem boils down to solving the heat equation with a free boundary, which is determined by a matching condition. The main equations for the supercooled Stefan problem are very similar to the mean-field banking equations:

$$\begin{aligned} &\frac{\partial}{\partial t}p(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x), \quad b(t) \le x < \infty, \\ &p(0,x) = \delta_z(x), \quad p(t,b(t)) = 0, \quad p(t, X \to \infty) \to 0, \end{aligned} \tag{104}$$

where p is the negative temperature profile, and b is the liquid–solid boundary. The location of the boundary is determined by the matching condition

$$\frac{d}{dt}b(t) = \frac{\alpha}{2}\frac{\partial}{\partial x}p(t,x). \tag{105}$$

As usual, we represent p as $p = H + r$, where r solves the following IBVP:

$$\begin{aligned} &\frac{\partial}{\partial t}r(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}r(t,x), \quad b(t) \le x < \infty, \\ &r(0,x) = 0, \quad r(t,b(t)) = -H(t,b(t)-z), \quad r(t, X \to \infty) \to 0. \end{aligned} \tag{106}$$

By using Eq. (20), we get the following system of coupled Volterra equations:

$$\begin{aligned} &\nu(t') + \int_0^t \frac{\Theta(t,t')\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')}}dt' + H(t,b(t)-z) = 0, \\ &b(t) + \frac{\alpha}{2}\int_0^t \frac{\Xi(t,t')\nu(t')}{\sqrt{2\pi(t-t')}}dt' = 0, \end{aligned} \tag{107}$$

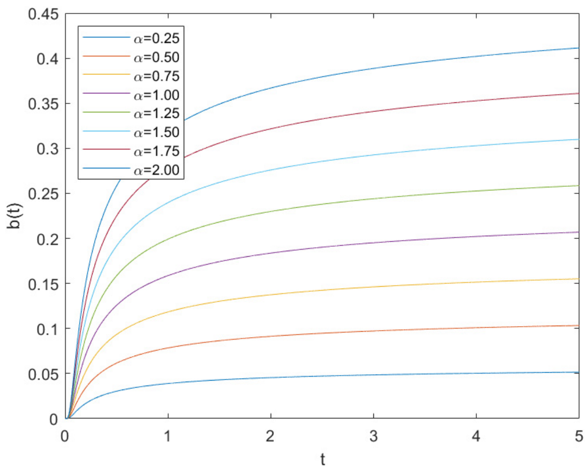


Fig. 8. Here we show the solid–liquid boundaries $b(t)$ for several representative values of α.

where

$$\Theta(t,t') = \frac{b(t)-b(t')}{(t-t')}, \quad \Xi(t,t') = e^{-\frac{(t-t')\Theta^2(t,t')}{2}}, \quad \Theta(t,t) = \frac{db(t)}{dt}, \quad \Xi(t,t) = 1. \tag{108}$$

System of integral equations (107) is very similar to the system (54) and can be solved by the same token.

In Fig. 8, we show $b(t)$ for several representative values of α.

7. The Integrate-and-Fire Neuron Excitation Model

7.1. *Governing equations*

We briefly describe the famous integrate-and-fire model in neuroscience, see e.g. Lewis & Rinzel (2003), Ostojic *et al.* (2009) and Carrillo *et al.* (2013). The integrate-and-fire model is a mathematical description of the properties of specific cells (spiking neurons) in the nervous system generating sharp electrical potentials across their cell membrane. These spikes last roughly for 1 ms. Spiking neurons are a significant signaling unit of the nervous system as a whole, so understanding their operation is of great theoretical and practical importance.

The neuron excitation problem has the form

$$\begin{aligned}
&\frac{\partial}{\partial t}p(t,x) = \frac{\partial}{\partial x}((x-\mu(t))p(t,x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x) + \lambda(t)\delta_{X_0}(x), \quad -\infty < x \le 0,\\
&p(0,x) = p_0(x), \quad p(t,-\infty) = 0, \quad p(t,0) = 0,\\
&X_0 < 0, \quad \lambda(t) = -\frac{1}{2}\frac{\partial}{\partial x}p(t,0), \quad \mu(t) = m_0 + m_1\lambda(t),
\end{aligned} \tag{109}$$

where $p(t,x) > 0$ is the probability density of finding neurons at a voltage x. Without loss of generality, we choose

$$p_0(x) = \delta_\xi(x), \quad \xi < 0. \tag{110}$$

Equations (109) preserve probability in the sense that

$$\frac{d}{dt}\int_{-\infty}^{0} p(t,x)dx = 0. \tag{111}$$

Indeed, integration of the main equation yields

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{0} p(t,x)dx &= \int_{-\infty}^{0} \frac{\partial}{\partial t}p(t,x)dx\\
&= \int_{-\infty}^{0}\left(\frac{\partial}{\partial x}((x-\mu(t))p(t,x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}p(t,x) + \lambda(t)\delta_{X_0}(x)\right)dx\\
&= \frac{1}{2}\frac{\partial}{\partial x}p(t,0) + \lambda(t) = 0.
\end{aligned} \tag{112}$$

7.2. The stationary problem

Because the integrate-and-fire equations are probability-preserving, there exists a stationary solution, which solves the time-independent Fokker–Planck problem

$$\begin{aligned} &0 = \frac{\partial}{\partial x}((x-\mu)p(x)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}p(x) + \lambda\delta_{X_0}(x), \\ &p(-\infty) = 0, \quad p(0) = 0, \quad -\infty < x \leq 0, \\ &\lambda = -\frac{1}{2}\frac{\partial}{\partial x}p(0), \quad \mu = m_0 + m_1\lambda. \end{aligned} \tag{113}$$

We represent $p(x)$ in the form

$$p(x) = p^{<}(x)(1-\Theta(x-X_0)) + p^{>}(x)\Theta(x-X_0), \tag{114}$$

where $\Theta(\cdot)$ is the Heaviside function, and notice that

$$\begin{aligned} &p^{<}(X_0) = p^{>}(X_0) \equiv \nu, \\ &\frac{1}{2}\left(\frac{\partial}{\partial x}p^{>}(X_0) - \frac{\partial}{\partial x}p^{<}(X_0)\right) = -\lambda, \end{aligned} \tag{115}$$

where ν, λ are unknown constants, which have to be determined as part of the solution. In view of the boundary conditions, it is clear that

$$\begin{aligned} &(x-\mu)p^{>}(x) + \frac{1}{2}\frac{\partial}{\partial x}p^{>}(x) = c^{>} \equiv -\lambda, \\ &(x-\mu)p^{<}(x) + \frac{1}{2}\frac{\partial}{\partial x}p^{<}(x) = c^{<} \equiv 0. \end{aligned} \tag{116}$$

Moreover, since p is continuous at $x = X_0$, the second matching condition (115) is satisfied automatically.

The method of separation of variables yields

$$p^{<}(x) = \nu e^{(X_0-\mu)^2-(x-\mu)^2}, \tag{117}$$

while the method of variation of constants yields

$$p^{>}(x) = 2\lambda(e^{\mu^2-(x-\mu)^2}D(-\mu) - D(x-\mu)), \tag{118}$$

where $D(\cdot)$ is Dawson's integral,

$$D(x) = e^{-x^2}\int_0^x e^{y^2}dy. \tag{119}$$

Thus,

$$\nu = 2\lambda(e^{\mu^2-(X_0-\mu)^2}D(-\mu) - D(X_0-\mu)), \tag{120}$$

and

$$p^{<}(x) = 2\lambda(e^{\mu^2}D(-\mu) - e^{(X_0-\mu)^2}D(X_0-\mu))e^{-(x-\mu)^2}. \tag{121}$$

At the same time, in the stationary case, the probability density $p(x)$ has to integrate to unity:

$$\int_{-\infty}^{0} p(x)dx = \int_{-\infty}^{X_0} p^{<}(x)dx + \int_{X_0}^{0} p^{>}(x)dx = 1, \tag{122}$$

which is a nonlinear equation for λ, because both μ and ν are known functions of λ. Once this equation is solved numerically, the entire profile is determined. It is worth noting that the integral $\int_{-\infty}^{X_0} p^{<}(x)dx$ can be computed analytically:

$$\begin{aligned}\int_{-\infty}^{X_0} p^{<}(x)dx &= \nu\int_{-\infty}^{X_0} e^{(X_0-\mu)^2-(x-\mu)^2}dx\\ &= \sqrt{\pi}\nu e^{(X_0-\mu)^2}N(\sqrt{2}(X_0-\mu)),\end{aligned} \tag{123}$$

while the second integral $\int_{X_0}^{0} p^{>}(x)dx$ can be split into two parts, the first of which can be computed analytically, and the second one has to be computed numerically:

$$\begin{aligned}&\int_{X_0}^{0} 2\lambda(e^{-x(x-2\mu)}D(-\mu) - D(x-\mu))dx\\ &\quad = 2\lambda\left(\sqrt{\pi}e^{\mu^2}(N(-\sqrt{2}\mu) - N(\sqrt{2}(X_0-\mu)))D(-\mu) - \int_{X_0-\mu}^{-\mu} D(x)dx\right).\end{aligned} \tag{124}$$

Thus, the corresponding nonlinear equation for λ can be written as

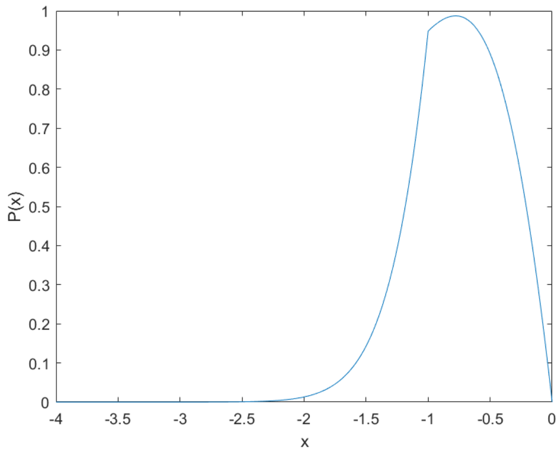
$$\begin{aligned}&\sqrt{\pi}(e^{\mu^2}N(-\sqrt{2}\mu)D(-\mu) - e^{(X_0-\mu)^2}N(\sqrt{2}(X_0-\mu))D(X_0-\mu))\\ &\quad - \int_{X_0-\mu}^{-\mu} D(x)dx - \frac{1}{2\lambda} = 0.\end{aligned} \tag{125}$$

We show the stationary profile $p(x)$ and its derivative $dp(x)/dx$ in Fig. 9. As expected, $dp(x)/dx$ jump down at $x = X_0$.

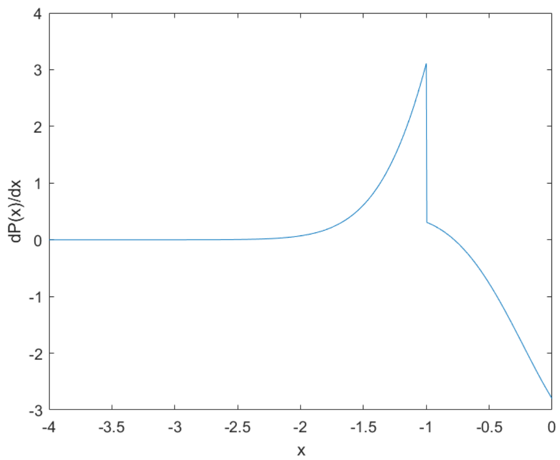
7.3. *The nonstationary problem*

First, we use the following transformation of variables:

$$\begin{aligned}&t = t, \quad y = x - M(t), \quad M(0) = 0,\\ &\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - M'(t)\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y},\end{aligned} \tag{126}$$



(a)



(b)

Fig. 9. In panels (a) and (b), we show the stationary distribution $p(x)$ and its derivative $dp(x)/dx$ for the following parameter values: $X_0 = -1$, $m_0 = 0.5$, and $m_1 = 0.1$. The corresponding value of λ, which is computed as part of the solution, is 1.4002.

and get the following IBVP:

$$
\begin{aligned}
\frac{\partial}{\partial t}p(t,y) &= \frac{\partial}{\partial y}((y + M'(t) + M(t) - \mu(t))p(t,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}p(t,y) \\
&\quad + \lambda(t)\delta_{X_0 - M(t)}(x), \quad \infty < y \le -M(t), \\
p(0,y) &= \delta_\xi(y), \quad p(t,-\infty) = 0, \quad p(t,-M(t)) = 0, \\
\lambda(t) &= -\frac{1}{2}\frac{\partial}{\partial y}p(t,-M(t)), \quad \mu(t) = m_0 + m_1\lambda(t).
\end{aligned}
\tag{127}
$$

Thus, by choosing M in such a way that

$$M'(t) + M(t) - \mu(t) = 0, \quad M(0) = 0, \tag{128}$$

or, explicitly,

$$M(t) = \int_0^t e^{-(t-t')}\mu(t')dt', \tag{129}$$

we get the IBVP for the standard Ornstein–Uhlenbeck process:

$$\begin{aligned}
&\frac{\partial}{\partial t}p(t,y) = \frac{\partial}{\partial y}(yp(t,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}p(t,y) + \lambda(t)\delta_{X_0-M(t)}(x), \quad \infty < y \le -M(t),\\
&\quad p(0,y) = \delta_\xi(y), \quad p(t,-\infty) = 0, \quad p(t,-M(t)) = 0,\\
&\quad \lambda(t) = -\frac{1}{2}\frac{\partial}{\partial y}p(t,-M(t)), \quad \mu(t) = m_0 + m_1\lambda(t).
\end{aligned} \tag{130}$$

As usual, we split $p(t,x)$ as follows:

$$p(t,x) = e^t H(\eta(t), e^t y - \xi) + r(t,x), \tag{131}$$

where the first term solves the governing equation and satisfies the initial, but not the boundary conditions, while $r(t,x)$ solves the following IBVP:

$$\begin{aligned}
&\frac{\partial}{\partial t}r(t,y) = \frac{\partial}{\partial y}(yr(t,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}r(t,y) + \lambda(t)\delta_{X_0-M(t)}(x), \quad \infty < y \le -M(t),\\
&\quad r(0,y) = 0, \quad r(t,-\infty) = 0, \quad r(t,-M(t)) = \chi_0(t),\\
&\quad \lambda(t) = -\frac{1}{2}\frac{\partial}{\partial y}r(t,-M(t)) + \chi_1(t), \quad \mu(t) = m_0 + m_1\lambda(t),
\end{aligned} \tag{132}$$

where

$$\begin{aligned}
\chi_0(t) &= -e^t H(\eta(t), e^t M(t) + \xi),\\
\chi_1(t) &= -\frac{e^{2t}(e^t M(t) + \xi)}{2\eta(t)} H(\eta(t), e^t M(t) + \xi).
\end{aligned} \tag{133}$$

We apply the familiar change of variables (88) and get the following IBVP for $q(\tau,\theta) = e^{-t}r(t,y)$:

$$\begin{aligned}
&\frac{\partial}{\partial \tau}q(\tau,\theta) = \frac{1}{2}\frac{\partial^2}{\partial \theta^2}q(t,\theta) + \varkappa(\tau)\delta_{X_0-M(t)}(x), \quad \infty < \theta \le \Gamma(\tau),\\
&\quad q(0,\theta) = 0, \quad q(\tau,-\infty) = 0, \quad q(\tau,\Gamma(\tau)) = \varrho_0(\tau),\\
&\quad \varkappa(\tau) = -\frac{(1+2\tau)}{2}\frac{\partial}{\partial \theta}q(\tau,\Gamma(\tau)) + \varrho_1(\tau),
\end{aligned} \tag{134}$$

where

$$\begin{aligned}\Gamma_0(\tau) &= \sqrt{1+2\tau}(X_0 - M(\ln(\sqrt{1+2\tau}))),\\ \Gamma(\tau) &= -\sqrt{1+2\tau}M(\ln(\sqrt{1+2\tau})),\\ \varrho_0(\tau) &= -H(\tau,\Gamma(\tau)-\xi),\\ \varrho_1(\tau) &= \frac{(\Gamma(\tau)-\xi)}{2\tau}H(\eta(t),\Gamma(\tau)-\xi).\end{aligned} \tag{135}$$

Denoting $q(\tau,\Gamma_0(\tau))$ by $\nu(\tau)$, we can split the IBVP (134) into two IBVPs:

$$\begin{aligned}&\frac{\partial}{\partial\tau}q^{>}(\tau,\theta) = \frac{1}{2}\frac{\partial^2}{\partial\theta^2}q^{>}(t,\theta), \quad \Gamma_0(\tau)\le\theta\le\Gamma(\tau),\\ &q^{>}(0,\theta)=0, \quad q(\tau,\Gamma_0(\tau))=\nu(\tau), \quad q(\tau,\Gamma(\tau))=\varrho_0(\tau),\end{aligned} \tag{136}$$

$$\begin{aligned}&\frac{\partial}{\partial\tau}q^{<}(\tau,\theta) = \frac{1}{2}\frac{\partial^2}{\partial\theta^2}q^{<}(t,\theta), \quad \infty<\theta\le\Gamma_0(\tau),\\ &q^{<}(0,\theta)=0, \quad q(\tau,\theta\to-\infty)\to 0, \quad q(\tau,\Gamma_0(\tau))=\nu(\tau),\end{aligned} \tag{137}$$

and a matching condition:

$$\frac{\partial}{\partial\theta}q^{<}(t,\Gamma_0(\tau)) - \frac{\partial}{\partial\theta}q^{>}(t,\Gamma_0(\tau)) = 2\left(-\frac{(1+2\tau)}{2}\frac{\partial}{\partial\theta}q^{>}(\tau,\Gamma(\tau))+\varrho_1(\tau)\right). \tag{138}$$

We can now use the results from Sec. 2 to reduce these equations to a very efficient (but highly nonlinear) system of Volterra integral equations. An analysis of the corresponding system will be presented elsewhere.

8. Conclusions

In this paper, we have described an analytical framework for solving several relevant and exciting problems of financial engineering. We have shown that the EMHP is a powerful tool for reducing partial differential equations to integral equations of Volterra type. Due to their unique nature, these equations are relatively easy to solve. In some cases, we can solve these equations analytically by judiciously using the Laplace transform. In other cases, we can solve them numerically by constricting highly accurate numerical quadratures. We have demonstrated that the EMHP has numerous applications in mathematical finance and far beyond its confines.

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